Contribution to the study of aging in disordered systems

Doctoral Thesis

submitted to

the Mathematisch-Naturwissenschaftlichen Fakultät of

the Rheinische Friedrich-Wilhelms-Universität Bonn

and

the Faculté des sciences of

the Aix-Marseille Université

presented by

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Bonn and Marseille

December 2013

With the permission of the Mathematisch-Naturwissenschaftliche Fakultät of the Rheinischen Friedrich-Wilhelms-Universität Bonn and of the Faculté des sciences of the Aix-Marseille Université.

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Date of the defense: 28. 03. 2014
Year of publication: 2014

Abstract

In this thesis we study the long-time behavior of random motions in random environments. More precisely, the focus lies on identifying general mechanisms that lead to specific long-time behavior, known in the physics literature as *aging*. Aging is a property of random motions whose convergence towards equilibrium becomes slower as time elapses. This becomes apparent in the fact that certain *time-time correlation functions* that measure the correlation between the motion at two points in time, t_w and $t_w + t$ are never independent of the age of the system, t_w. A universal approach to this problem was developed over the past decades: the long-time behavior of correlation functions can be linked to the long-time behavior of the *clock process*, which is the total time elapsed along a given length of the trajectory of the random motion.

Recently, a very elegant approach to studying clock processes was proposed by Gayrard in [37, 36]. Here, the clock process is viewed as a partial sum process whose increments are dependent random variables and then convergence criteria, due to Durrett and Resnick, [33], are employed to prove convergence to a subordinator. This method was further developed by Bovier and Gayrard in [28], where sufficient conditions for convergence of clock processes with strongly dependent increments are established for random motions on sequences of finite graphs.

The purpose of this thesis is to extend the methods that were developed in [37] and [28] for proving convergence of clock processes and to use these methods to study the aging behavior of various models. The approach in this thesis also is based on [33] and the results can be divided into three distinct parts. In the first part we establish criteria for the convergence of clock processes on sequences of finite graphs to *extremal processes*. Extremal processes are constructed from the maximal jumps of α-stable subordinators and it is known, [44], that they arise as limits of α-stable subordinators in the limit as $\alpha \to 0$. We use these criteria to prove that the results on so-called *extremal aging* for p-spin SK models that were obtained in law with respect to the environment by Ben Arous and Gün in [19] hold almost surely for $p > 4$, respectively in probability for $2 \leq p \leq 4$. The second part of the thesis deals with random motions that are defined on *infinite graphs*. More precisely, we introduce sufficient conditions for the underlying clock process to converge to a subordinator without drift. We then use these conditions to establish the existence of a *normal aging* regime in Bouchaud's asymmetric trap model on $\mathbb{Z}^d$, for $d \geq 2$, extending previous results due to Barlow and Černý, [2, 31] [53]. In the third part of this thesis we consider Bouchaud's asymmetric trap model on $\mathbb{Z}^d$, $d \geq 3$, and its symmetric version on $\mathbb{Z}^2$. We study the effect of a class of initial distributions on the behavior of correlation functions and prove the existence of an *super-aging* regime.

Zusammenfassung

In dieser Dissertation wird das Langzeitverhalten zufälliger Bewegungen in zufälligen Umgebungen untersucht. Der Fokus liegt auf der Identifizierung allgemeiner Mechanismen, die ein spezielles Langzeitverhalten dieser Bewegungen erklären, welches in der Physikliteratur als *Altern* bekannt ist. Altern bedeutet in diesem Zusammenhang, dass die Konvergenzgeschwindigkeit der zufälligen Bewegung zum Gleichgewicht mit steigender Zeit langsamer wird. Dies ist daran sichtbar, dass gewisse *Zeit-Zeit Korrelationsfunktionen*, welche die Korrelation der Bewegung zu zwei verschiedenen Zeitpunkten t_w and $t_w + t$ messen, stets vom Alter des Systems, t_w, abhängen. Über die letzten Jahrzehnte entwickelte sich ein allgemeiner Zugang zu diesem Problem: Es gibt einen direkten Zusammenhang zwischen dem Langzeitverhalten der Korrelationsfunktionen und dem Langzeitverhalten des *Uhrenprozesses* der zufälligen Bewegung, wobei der Uhrenprozess die gesamte Zeit misst, die entlang einer gegebenen Länge der Trajektorie verstrichen ist.

In [37, 36] wurde von Gayrard ein sehr eleganter Ansatz vorgeschlagen um das Konvergenzverhalten von Uhrenprozessen zu studieren. Dort wird der Uhrenprozess als ein Partialsummenprozess mit zufälligen und *abhängigen* Zuwächsen betrachtet, für den Konvergenzkriterien aus [33] angewendet werden können um die Konvergenz gegen einen Subordinator zu erhalten. Diese Methode wurde von Bovier und Gayrard in [28] weiterentwickelt. Die dort erstellten hinreichenden Bedingungen für die Konvergenz von Uhrenprozessen mit höchst korrelierten Zuwächsen wurden im Hinblick auf das Konvergenzverhalten zufälliger Bewegungen, die auf einer Folge von endlichen Graphen definiert sind, erstellt.

In der vorliegenden Doktorarbeit erweitern wir die Methoden, die in [37] und [28] für Konvergenzbeweise von Uhrenprozessen entwickelt wurden und benutzen diese um das Alterungsverhalten weiterer Modelle zu untersuchen. Die Resultate dieser Arbeit sind in drei Teile unterteilt und der Ansatz, der in dieser Arbeit verfolgt wird, baut ebenso auf [33] auf. Der erste Teil beinhaltet Kriterien unter welchen Uhrenprozesse, die auf einer Folge von endlichen Graphen definiert sind, gegen *Extremalprozesse* konvergieren. Extremalprozesse werden aus den größten Sprüngen von α-stabilen Subordinatoren konstruiert, genauer gesagt können sie als Grenzwerte von α-stabilen Subordinatoren für $\alpha \to 0$ gesehen werden (vgl. [44]). Wir benutzen diese Kriterien um zu beweisen, dass die Resultate, die von Ben Arous und Gün in [19] zum *extremalen Altern* in p-spin SK Modellen in Verteilung bezüglich der zufälligen Umgebung bewiesen wurden, für $p > 4$ fast sicher, und für $2 \leq p \leq 4$ in Wahrscheinlichkeit gelten. Der zweite Teil dieser Arbeit behandelt zufällige Bewegungen, die auf *unendlichen Graphen* definiert sind. Insbesondere führen wir Bedingungen an den Uhrenprozess ein, die für die Konvergenz gegen einen Subordinator hinreichend sind. Mit Hilfe dieser Bedingungen beweisen wir die Existenz eines *normalen Alterungsregimes* im Bouchaudschen asymmetrischen Fallenmodell auf $\mathbb{Z}^d$, für $d \geq 2$. Dies erweitert bisherige Resultate von Barlow and Černý, [2, 31], und Mourrat [53]. Im dritten Teil der Arbeit studieren wir Bouchaud's asymmetrisches Fallenmodell auf $\mathbb{Z}^d$, $d \geq 3$, beziehungsweise Bouchaud's symmetrisches Fallenmodell auf $\mathbb{Z}^2$. Wir untersuchen den Einfluss einer Klasse von Anfangsverteilungen auf das Verhalten von Korrelationsfunktionen und beweisen die Existenz eines *Super-Alterungsregimes*.

Résumé

Dans cette thèse nous étudions le comportement en temps long de dynamiques en environnements aléatoires. Plus précisément, nous cherchons à identifier des mécanismes généraux qui sont à l'origine d'un comportement spécifique en temps long, connu dans la littérature physique sous le nom de vieillissement. Le vieillissement est une propriété des dynamiques dont la convergence vers l'équilibre ralentit au cours du temps. Cela s'observe dans le comportement de certaines fonctions de corrélation temps-temps, qui ne deviennent jamais indépendantes de l'âge du système, c'est-à-dire la corrélation entre les états de la dynamique à deux instants donnés t_w et $t_w + t$ ne devient jamais indépendante de t_w. Une approche universelle à ce problème fut développée durant les dernières décennies: le comportement en temps long des fonctions de corrélation peut être lié à celui du *processus d'horloge*, qui est le temps total écoulé le long d'une trajectoire de longueur fixée de la dynamique.

Récemment, une approche très élégante fut proposée par Gayrard [37, 36] pour étudier le processus d'horloge. Celui-ci est vu comme un processus de sommes partielles à incréments corrélés auquel des critères de convergence, dûs à Durett et Resnick [33], sont appliqués pour en déduire la convergence vers un subordinateur. Cette méthode fut poussée plus avant par Bovier et Gayrard qui donnent en [28] des conditions suffisantes de convergence pour des processus d'horloge à incréments fortement corrélés dans le cas de dynamiques aléatoires sur des suites de graphes finis.

Dans cette thèse, nous étendons les méthodes développées dans [37] et [28] pour prouver la convergence de processus d'horloge, et étudions le phénomène de vieillissement dans divers modèles. Notre approche est basée sur [33] et nos résultats peuvent être divisés en trois parties distinctes. Dans la première partie, nous établissons des critères de convergence vers des *processus extrémaux* pour des processus d'horloge sur des graphes finis. Les processus extrémaux sont construits à partir des sauts maximaux de subordinateurs α-stables et ils s'obtiennent [44] comme limites de ces subordinateurs lorsque $\alpha \to 0$. Nous utilisons ensuite ces critères pour prouver que les résulats obtenus en loi par Ben Arous et Gün [19] sur le *vieillissement extrémal* du modèle du SK à p-spins sont vrais presque sûrement si $p > 4$ et ou en probabilité si $2 \leq p \leq 4$. La deuxième partie de la thèse traite de dynamiques sur des graphes infinis. Nous donnons des conditions suffisantes sous lesquelles le processus d'horloge sous-jacent converge vers un subordinateur, et utilisons ensuite ces conditions pour établir l'existence d'un régime de *vieillissement normal* dans le modèle assymétrique de pièges de Bouchaud sur $\mathbb{Z}^d$ pour $d \geq 2$. Ceci constitue une extension des résultats de Barlow et Černý [2, 31] et Mourrat [53]. La troisième partie concerne le modèle de Bouchaud assymétrique lorsque $d \geq 3$ et sa version symétrique lorsque $d = 2$. Nous étudions l'effet d'une classe de distributions initiales sur le comportement des fonctions de corrélation et prouvons l'existence d'un régime de *sur-vieillissement*.

Acknowledgement

First of all my thanks go to my advisors Anton Bovier and Véronique Gayrard for giving me the opportunity to undertake the doctoral studies, for having confidence in me, and for all their help and support. They gave me many advice and supported my career. Working with them was a great experience and I will never forget my PhD time.

Taking this opportunity, I would like to thank Erwin Bolthausen, Jiří Černý, Herbert Koch, Corinna Kollath, and Pierre Picco either for being part of my jury or being a referee of my thesis. I am grateful to Dmitry Ioffe and Oren Louidor for the opportunity to work as a Postdoc at the Technion. Finally, I thank Benjamin Schlein for being my mentor.

It was a pleasure to be part of the probability theory (and stochastic analysis) group at the University of Bonn. In particular, I would like to thank the Postdocs Sebastian, Giada, Loren, Nicola, and Evangelia for their support and advice. Many thanks also go to the PhD students Martina, René, Carina, Daniel, Lisa, Hannah, Patrick, Peter, Rebecca, Nikolaus, Martin, Shidong, and all the other participants of the cake seminar, as well as to Mei-Ling.

Special thanks go to my family for their endless support: Děkuji vám mockrát! Also, I am very grateful for all the help I received from Steffen's family.

Without my great friends, the last couple of years would have been very poor!

Last, but not least, I sincerely thank Steffen for his infinite patience and support. Thanks for always having my back!

Financial support through the Bonn International Graduate School, through the Collaborative Research Center 1060, and through Aix-Marseille University is gratefully acknowledged.

Contents

Chapter 1

Introduction

This thesis contains mathematical analysis of the aging behavior of disordered systems such as spin glasses. Let us describe this phenomenon with the help of the following experiment: A material is cooled below its glass transition temperature in a magnetic field. This field is kept until a waiting time t_w and then removed. At t_w the material is magnetized and the magnetization starts to decrease after t_w. The quantity one is now interested in is the decrease of the magnetization, measured at two points in time, $(t_w, t_w + t)$. When performing this experiment on a ferromagnet, one observes that the decrease of magnetization depends for large t_w only on the length of this interval, t. This observation is not at all true when the experiment is performed on a *spin glass*: the decrease of magnetization never becomes independent of the *age* of the system, t_w. It always depends on t_w and t in such a way that the older the system gets, the longer it takes to forget its past. In other words, the larger t_w the larger one must choose t to see a significant decrease of magnetization. This phenomenon of *dynamics of spin glasses* is called *aging*.

Several approaches and models were proposed by theoretical physicists to explain this peculiar behavior, the main difference between them being the time scale on which the dynamics is observed in relation to the volume of the spin glass; see Chapter VI of [50] and [25] for an overview. On the one hand, in *activated aging*, one assumes that the time scale on which the dynamics is observed diverge with the size of the volume and the size of the volume tends to infinity. The intuitive picture is that the dynamics is crossing 'thermally activated barriers'. This was first proposed by Goldstein in [41]. On the other hand, in *non-activated aging*, one studies directly infinite systems at large times. How to implement these ideas and how to derive information on the aging behavior of dynamics of spin glasses turns out to be a very challenging task - even on a non-rigorous level. In this respect the introduction of simpler phenomenological models was of great importance. The most prominent models of this type are the so-called trap models and were among others introduced and studied by Bouchaud, Dean, Maass, Monthus, Rinn (see [24, 26, 51, 58, 59]). In these models, the dynamics of a spin glass is described as a particle that evolves on a landscape consisting of random 'traps'. These traps represent the states in which the dynamics spends most of its time. For simplicity, one typically also assumes that the traps are independent of each other.

In such simple models the aging phenomenon can be captured by considering the event that the dynamics does not exit the state it was in at t_w during the time interval $(t_w, t_w + t)$. Therefore, to gain information on their aging behavior, one studies time-time correlation functions (often also called two-point, or two-time correlation functions) that measure the dependence of the position of the particle at two distinct time points t_w and $t_w + t$.

Let us explain the idea of such trap models in the context of *Bouchaud's trap model on the complete graph*. The simplest physical models of dynamics are *Glauber dynamics* of mean-field spin glasses such as Derrida's random energy model (hereafter REM), [32], or the p-spin Sherrington Kirkpatrick models (hereafter p-spin SK models), [60]. Here, the configuration space is the n

dimensional hypercube, that is $\Sigma_n = \{-1,1\}^n$. The energy landscape is given by a random function $H_n(x)$, where $H_n = \{H_n(x), x \in \Sigma_n\}$ is a Gaussian process with zero mean and correlation structure,

$$\mathbb{E}H_n(x)H_n(x') = nR_n(x,x')^p, \tag{1.0.1}$$

where $p \geq 2$, and where

$$R_n(x,x') \equiv 1 - \tfrac{2\,\mathrm{dist}(x,x')}{n} = 1 - n^{-1}\textstyle\sum_{i=1}^n |x_i - x_i'|, \tag{1.0.2}$$

is the so-called overlap. The two extremal cases with respect to p are: $p = 2$ (the SK model), where the correlation is the strongest, and '$p = \infty$' (the REM) where H_n is a sequence of i.i.d. random variables. Glauber dynamics are Markov jump processes, X_n, that are reversible with respect to the measure that assigns to $x \in \Sigma_n$ a mass proportional to $\tau_n(x) = \exp(-\beta H_n(x))$, where $\beta > 0$ is the inverse temperature. More precisely, X_n jumps between two sites $x \in \Sigma_n$ and $y \in \Sigma_n$ according to transition probabilities $p_n(x,y)$ that satisfy the detailed balance condition

$$\tau_n(x)p_n(x,y) = \tau_n(y)p_n(y,x), \quad x,y \in \Sigma_n, \tag{1.0.3}$$

and stays in $x \in \Sigma_n$ an exponential time with mean proportional to $\tau_n(x)$. The idea of simplification in Bouchaud's trap model on the complete graph is to say that it only can visit those x for which $\tau_n(x)$ is maximal. Since H_n is a Gaussian process indexed by Σ_n, there are $n \sim \log|\Sigma_n|$ such values and they are of the order of $\exp(\sqrt{2\log 2}n + E_i)$, where $\{E_i, 1 \leq i \leq n\}$ are exponential random variables. Hence, the relevant part of the graph can be represented on $\{1,\dots,n\}$ and the relevant energy landscape is given by E_i, where $\{E_i, 1 \leq i \leq n\}$ is a collection of i.i.d. exponential random variable having mean one. Now, Bouchaud's trap model on the complete graph is the process $\widetilde{X}_n$ with state space $\{1,\dots,n\}$ and that jumps from $i \in \{1,\dots,n\}$ to $j \in \{1,\dots,n\}$ uniformly at random. It stays in a site i an exponential time with mean value $\tau_n(i) = \exp(\beta E_i)$, and so

$$\mathbb{P}(\tau_n(i) > u) = \begin{cases} u^{-1/\beta} & , \quad \text{if} \quad u \geq 1 \\ 1 & , \quad \text{if} \quad u < 1 \end{cases}, \quad 1/\beta \in (0,1), \quad i \in \{1,\dots,n\}. \tag{1.0.4}$$

In order to study the aging behavior of $\widetilde{X}_n$, one chooses a time scale c_n on which the process is observed. Then, aging is measured in the behavior of correlation functions such as the probability that $\widetilde{X}_n$ did not jump during the time interval $(c_n t_w, c_n(t_w+t))$ or the probability that $\widetilde{X}_n(c_n t_w) = \widetilde{X}_n(c_n(t_w+t))$. As explained in [24], these correlation functions are, for large n, functions of the ratio t_w/t for $\widetilde{X}_n$.

The intention behind introducing Bouchaud's trap model on the complete graph was to construct a 'toy model' to understand aging behavior of the REM or the p-spin SK models. However, only roughly 20 years later, it was rigorously established that Bouchaud's trap model on the complete graph mimics the correct aging behavior for mean-field spin glass models. First, this was done for the so-called *random hopping dynamics* (see Section 1.3 for its definition) of the REM on time scales close to equilibrium by Ben Arous, Bovier, and Gayrard in [9, 10, 12]. Later, the same model was studied on time scales that are far from equilibrium and are beyond Bouchaud's heuristics by Ben Arous and Černý in [17]. It was therefore surprising that the same function as Bouchaud predicted was found in the description of its aging behavior. Finally, the full time and temperature domain in which aging occurs was investigated by Gayrard in [36]. Moreover, the p-spin SK models were proven to present on the same time scales as in [17] the same aging behavior as Bouchaud's trap model on the complete graph, first in law with respect to the random environment by Ben Arous, Bovier, and Černý in[8] and then almost surely for $p > 4$, respectively in probability for $p = 3, 4$, by Bovier and Gayrard in [28]. All these results show that Bouchaud's trap model (on the complete graph) is a good ansatz and in fact suggest that there should be a *universality class* in which Bouchaud's picture should be relevant.

Trap models were introduced and studied on various graphs, among others on the d-dimensional lattice, $\mathbb{Z}^d$, by Bouchaud, Maass, and Rinn in [58, 59]. These trap models were also defined for asymmetric dynamics, where the particle does not jump uniformly at random among its neighbors but its jump direction is also influenced by the τ_n's (see Section 1.4 for a definition). The mathematical community became interested into these models in the beginning of the 21^{st} century. The first rigorous analysis was carried by Fontes, Isopi, and Newman on Bouchaud's trap model on $\mathbb{Z}$ in [35]. Then, Ben Arous and Černý proved in [14] aging in the asymmetric version of this model. Later, the higher dimensional versions of the symmetric model were considered: first for $d = 2$ by Ben Arous, Černý, and Mountford in [18] and then for $d \geq 3$ by Ben Arous and Černý in [16].

One common characteristics in all the above mentioned models is the recurrent appearance of the arcsine law of α-stable subordinators, $\alpha \in (0, 1)$, in the description of their aging behavior which suggests a wide universality class. This universality was explained for the first time for a class of dynamics, the so-called random hopping dynamics (see Section 1.3 for its definition), by Ben Arous and Černý in [15]. There, it is established how the long-time behavior of correlation functions can be derived from the long-time behavior of *clock processes* - the total time elapsed along a given length of the trajectory of the random motion. An elegant approach to prove convergence of clock processes is presented for general reversible Markov jump processes by Gayrard in [37]. The idea in [37] is to apply convergence criteria for partial sum processes due to Durrett and Resnick, [33], to prove convergence of the clock process to a subordinator. This method was further adapted in the setting when the dynamics is defined on a sequence of finite graphs and when the increments of the clock process are very correlated by Bovier and Gayrard in [28]. In this thesis, we resume the concept put forward in [37] and [28] and develop further techniques for proving convergence of clock processes based on [33]. Specifically, we extend the results of [37] and [28] to the setting when α tends to zero and to dynamics that are defined on infinite graphs.

Before going into further discussion of the results of this thesis, let us describe the general setting we consider, the key quantities we study, and the main questions we ask. This is the content of the following three sections. Finally, in Section 1.4 we give a description of the main results of this thesis.

1.1 Markov jump processes in random environments, clock processes, and aging

Let us give a a formal definition of all dynamics we will consider. Let $G_n(\mathcal{V}_n, \mathcal{L}_n)$ be a sequence of loop-free graphs with set of vertices $\mathcal{V}_n$ and set of edges $\mathcal{L}_n$ (possibly $G_n(\mathcal{V}_n, \mathcal{L}_n) = G_\infty(\mathcal{V}_\infty, \mathcal{L}_\infty)$ for all n). We always assume that $|\mathcal{V}_n| \to \infty$ as $n \to \infty$. The *random environment* is a sequence of families of positive random variables, $\{\tau_n(x),\ x \in \mathcal{V}_n\}$, $n \in \mathbb{N}$, defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Note that we do not make any assumptions on the correlation structure of the random environment at this point. The process we are interested in, X_n, is constructed as follows. On $\mathcal{V}_n$ we consider an irreducible discrete time Markov chain J_n with transition probabilities $p_n(x, y)$, which might be random in the environment. We assume that J_n is reversible with respect to some measure π_n and denote the initial distribution of J_n by μ_n. Writing

$$\lambda_n(x) \equiv \pi_n(x)/\tau_n(x), \tag{1.1.1}$$

we define the sequence of clock processes as

$$\widetilde{S}_n(k) = \textstyle\sum_{i=0}^{k-1} \lambda_n^{-1}(J_n(i)) e_{n,i}, \quad k \in \mathbb{N}\ , \tag{1.1.2}$$

where $\{e_{n,i} : i \in \mathbb{N}_0, n \in \mathbb{N}\}$ is an array of independent and identically distributed exponential random variables with parameter one, independent of J_n and the random environment. Then, X_n, given by

$$X_n(t) = J_n(S_n^{\leftarrow}(t)), \quad \text{where} \quad , S_n^{\leftarrow}(t) = \inf\{k \geq 0 : S_n(k) \geq t\}, \tag{1.1.3}$$

is a continuous time Markov jump process with jump rates $\lambda_n(x, y) \equiv \lambda_n(x)p_n(x, y)$ and invariant measure that assigns to $x \in \mathcal{V}_n$ the mass $\tau_n(x)$. The behavior of X_n can be described as follows: at each vertex $x \in \mathcal{V}_n$, the process stays an exponential time with mean $\lambda_n(x)$ and then jumps to one of the neighbors, y, of x with probability $p_n(x, y)$. For further references, we write $\mathcal{P}_{\mu_n}$ for the law of X_n and P_{μ_n} for the law of J_n. Note that these distributions might be random in the probability space of the random environment $(\Omega, \mathcal{F}, \mathbb{P})$.

We are now ready to formally define the aging phenomenon. For this, we choose a time scale, c_n, on which we observe the process and a two-time correlation function, $\mathcal{C}_n(t, s)$, that measures the dependences of $X_n(c_n(t + s))$ on $X_n(c_n t)$.

Definition 1.1 (Definition 1.4 in [37])**.** *A time correlation function, $\mathcal{C}_n$, exhibits normal aging on the time scale c_n if one of the following relations is verified*

$$\lim_{n\to\infty} \mathcal{C}_n(t, t\rho) = \mathcal{C}_\infty(\rho), \qquad \rho \geq 0,\ t > 0 \quad \textit{arbitrary}, \tag{1.1.4}$$

$$\lim_{t\to\infty} \lim_{n\to\infty} \mathcal{C}_n(t, t\rho) = \mathcal{C}_\infty(\rho), \qquad \rho \geq 0, \tag{1.1.5}$$

where $\mathcal{C}_\infty : [0, \infty) \to [0, 1]$ is a non-trivial function, and for some convergence mode with respect to the probability law, $\mathbb{P}$, of the random environment.

The term normal aging refers to the fact that the decrease of correlation is observed on time intervals that are proportional to the age of the system. Other types of aging behavior were found, both in physical experiments and in mathematical models. Let us define the ones that appear in this thesis.

Definition 1.2. *If there exists $\delta \neq 1$ such that one of the relations* (1.1.4) *or* (1.1.5) *holds when we substitute $t^\delta \rho$ for $t\rho$, we say that $\mathcal{C}_n$ exhibits super-aging if $\delta < 1$, respectively subaging if $\delta > 1$.*

The term super(sub)-aging reflects the fact that the correlation of the process decreases only when it is observed on time intervals that are of larger (smaller) order than the age of the system.

This is to be contrasted with the following aging.

Definition 1.3. *If there exists a decreasing sequence α_n, $\alpha_n \to 0$, such that one of the relations* (1.1.4) *or* (1.1.5) *holds for $\mathcal{C}_n(t^{1/\alpha_n}, t^{1/\alpha_n}((1 + \rho)^{1/\alpha_n} - 1))$, we say that $\mathcal{C}_n$ exhibits extremal aging.*

The term extremal aging was first introduced in [19]. Here, the decrease of correlation is on time intervals of the form $[t^{1/\alpha_n}, (t + t\rho)^{1/\alpha_n}]$. Extremal aging corresponds to the setting when the dynamics is observed on very short time scales and hence correlation only decreases on time intervals that diverge with n.

The correlation functions one is typically interested in can be related to the probability that the process X_n did not jump during a time interval, say $(c_n t, c_n(t + s))$, or did not move 'too far' during that time interval. Since all Markov jump processes in random environments can be written using relations (1.1.2) and (1.1.3), the behavior of correlation functions is related to the study of

$$\mathcal{P}_{\mu_n}(\{\widetilde{S}_n(k), k \geq 0\} \cap (c_n t, c_n(t + s)) = \emptyset). \tag{1.1.6}$$

This highlights the importance of the clock process. Therefore, we seek for sequences, a_n, c_n, satisfying $a_n, c_n \to \infty$, as $n \to \infty$, such that S_n, given by

$$S_n(t) \equiv c_n^{-1} \widetilde{S}_n(\lfloor a_n t \rfloor) = c_n^{-1} \textstyle\sum_{i=0}^{\lfloor a_n t \rfloor - 1} \lambda_n^{-1}(J_n(i)) e_{n,i}, \quad t > 0, \tag{1.1.7}$$

converges in a suitable sense to a non-trivial limit. The sequences c_n are the time scale on which we observe the process X_n, while a_n is an auxiliary time scale for J_n. In some cases the clock process is diverging, respectively vanishing, faster, respectively slower, than a linear function so that there are no sequences a_n, c_n such that S_n has a non-trivial limit. In order to gain information on the convergence behavior of S_n, we then ask a more general question. Namely, we ask when there are sequences a_n, c_n and (non-linear functions) f_n, such that $f_n(S_n)$ converges in a suitable sense. Then, relation (1.1.3) and convergence of $f_n(S_n)$ allows us to study correlation functions that measure the dependence of $X_n(f_n^{-1}(c_n(t+s)))$ and $X_n(f_n^{-1}(c_n t))$.

It remains to find appropriate criteria for the clock process to converge and to draw consequences from the convergence of the clock process to correlation functions. The first step in this direction is to realize that for a fixed $\omega \in \Omega$, that is for a fixed realization of the random environment, the clock process is nothing but a sequence of partial sum process whose summands are positive random variables. The convergence behavior of partial sum processes is a classical question one asks in probability theory. Thus, let us first recall what is known form classical probability theory.

1.2 Convergence of partial sum processes

In this section, we state well-known convergence results for sequences of partial sum processes. Let us consider sequences of non-negative random variables, $Z = \{Z_i, i \geq 1\}$, that are defined on some abstract probability space $(\Omega, \mathcal{F}, \mathcal{P})$. For simplicity we assume that Z is a collection i.i.d. random variables. We are interested in the behavior of partial sum processes of these random variables. The first result in this direction is the law of large numbers and the central limit theorem for sums of random variables.

Theorem 1.4 (Law of large numbers and central limit theorem)**.** *Suppose that $\mathcal{E}Z_1 = \mu < \infty$. Then, $\mathcal{P}$-almost surely,*

$$\lim_{n\to\infty} n^{-1}\sum_{i=1}^{n} Z_i = \mu. \tag{1.2.1}$$

If moreover, $\mathcal{E}(Z_1^2) - (\mathcal{E}Z_1)^2 = \sigma^2 < \infty$, then we have in $\mathcal{P}$-law

$$\lim_{n\to\infty} (n\sigma^2)^{-1/2}\sum_{i=1}^{n}(Z_i - \mu) = X, \tag{1.2.2}$$

where $X \sim \mathcal{N}(0,1)$ is a standard normal random variable.

The following functional version of both statements is also well-known.

Theorem 1.5. *Suppose that $\mathcal{E}Z_1 = \mu < \infty$, then uniformly on compact intervals, $\mathcal{P}$-almost surely, the process*

$$\overline{S}_n(t) = n^{-1}\sum_{i=1}^{\lfloor nt\rfloor} Z_i, \tag{1.2.3}$$

converges to the deterministic process μt. If moreover, $\mathcal{E}(Z_1^2) - (\mathcal{E}Z_1)^2 = \sigma^2 < \infty$, then the process

$$(n\sigma^2)^{-1/2}\sum_{i=1}^{\lfloor nt\rfloor}(Z_i - \mu), \qquad t > 0, \tag{1.2.4}$$

converges in $\mathcal{P}$-law to a Brownian motion B. Convergence holds in $\mathcal{P}$-law in the space of continuous functions, $C[0,\infty)$, equipped with the topology of uniform convergence on compact intervals.

The normalization $n^{-1/2}$ in (1.2.2) and (1.2.4) comes from the fact that the maximal value of n random variables that admit second moments is at most of the order of $n^{1/2}$. In (1.2.1) and (1.2.3), the normalization n^{-1} arises since the maximum of n random variables having finite mean is smaller than n. Then, having normalized properly, no single value of the random variables $\{Z_1, \ldots, Z_n\}$ contributes substantially to the sum.

Naturally, the question arises what happens when Z_1 does not have a finite first moment. Then, the *extreme values* of $\{Z_1, \ldots, Z_n\}$ are of larger order than n and hence the behavior of sums and partial sum processes is dominated by the extreme values of the Z's. This brings *extreme value theory* into play and yields that different processes arise as limits.

Let us explain this in more detail. A key object that describes the structure of the Z_i's is the sequence of point processes,

$$\xi_n \equiv \sum_{i=1}^{n} \delta_{u_n^{-1}(Z_i)}, \quad n \in \mathbb{N}, \tag{1.2.5}$$

where u_n is a diverging sequence that normalizes the Z_i's. The following theorem is a classical convergence result for ξ_n.

Theorem 1.6 (Proposition 3.1 in [57]). *Let ξ be a Poisson random measure with intensity measure ν. Then, $\xi_n \Rightarrow \xi$, if and only if there exists an increasing sequence u_n such that, for all continuity points $u > 0$ of ν, $n\mathcal{P}(Z_i > u_n(u)) \to \nu(u, \infty)$. Convergence holds weakly in the space of point measures on* $(0, \infty)$.

Theorem 1.6 states that the collection of points $\{u_n^{-1}(Z_i), i \geq 1\}$ has asymptotically the same distribution as a *Poisson random measure*. Equipped with Theorem 1.6 one can now control partial sum process of the form

$$\overline{S}_n(t) \equiv \sum_{i=1}^{\lfloor nt \rfloor} u_n^{-1}(Z_i). \tag{1.2.6}$$

Theorem 1.6, and since the $u_n^{-1}(Z_i)$'s are the increments of $\overline{S}_n$, suggest that processes which arise as limits of partial sum processes have increments which are the points of a Poisson random measure. Such processes are called *subordinators* and are by construction strictly increasing. This is the content of the following theorem.

Theorem 1.7 (Proposition 3.4 in [57]). *Let V_α be an α-stable subordinator with Lévy measure $cu^{-\alpha}$, where $\alpha \in (0, 1)$, $c > 0$, and zero drift. Then, $\mathcal{P}(Z_i > n^{1/\alpha}u) \sim cx^{-\alpha}L(u)$, where L is a slowly varying function, that is when* $\lim_{u\to\infty} L(uv)/L(u) = 1$ *for all $v > 0$, if and only if $S_n \Rightarrow V_\alpha$. Convergence holds weakly in the space of càdlàg functions, $D[0, \infty)$, equipped with Skorohod's J_1 topology.*

The above theorem is the counterpart of the functional version of the law of large number in the setting when Z_1 has infinite mean.

One can also study the limiting case when $\alpha = 0$, that is when the tail of Z_1 is of the form $\mathcal{P}(Z_1 > u) \sim L(u)$ where L is again a slowly varying function. In this case, a non-linear transformation of partial sum processes is needed in order to obtain non-trivial limits. It is established in [45] that the correct transformation in this setting is given by

$$n^{-1}L(\overline{S}_n(t)) \equiv n^{-1}L\Big(\textstyle\sum_{i=1}^{\lfloor nt \rfloor} Z_i\Big), \quad t \geq 0. \tag{1.2.7}$$

Theorem 1.8 (Theorem 2.1 in [45]). *Suppose that $\mathcal{P}(Z_1 > u) \sim L(u)$ where L is a slowly varying function. Let ξ be a Poisson random measure on $(0, \infty) \times (0, \infty)$ with intensity measure $dt \times x^{-2}dx$ and let M be given by*

$$M(t) = \sup\{x_i : t_i \leq t\}, \quad t \geq 0. \tag{1.2.8}$$

Then, $n^{-1}L(\overline{S}_n) \Rightarrow M$. Convergence holds weakly in the space of càdlàg functions, $D[0,\infty)$, equipped with Skorokhod's J_1 topology.

Processes as M in Theorem 1.8 are called *extremal processes*. It is known, [44], that they arise as limits of α-stable subordinators when $\alpha \to 0$. Thus, Theorem 1.8 complements Theorem 1.7.

Finally, we obtain the following dichotomy: on the one hand, when the Z's have finite variance, then partial sum processes converge to *Gaussian* processes and on the other hand, when the Z's have infinite variance, partial sum processes converge to processes which in turn are constructed from *Poisson* random measures.

We have restated classical convergence results for sequences partial sum processes in the simplest setting, when the increments are i.i.d. random variables. A great deal of work also was carried out to extend the above mentioned theorems as stated by Durrett and Resnick: "for a variety of time scales and under a bewildering assortment of conditions" (page 829 in [33]). As we will see the broad validity of these results, in particular also for correlated sequences of arrays, is the source of universality.

1.3 Convergence of clock processes

In the previous section we have seen that stable subordinators arise as natural limits of sequences of partial sum processes when the increments have infinite variance. We explained this in the simplest possible setting, namely when the increments are i.i.d. random variables, which typically is not the case when studying clock processes. Specifically, the clock process is a sum of positive random variables that are dependent on each other because the summation is along trajectory of the chain J_n. Also, by construction (see (1.1.1) and (1.1.2)) the clock process is random in $(\Omega, \mathcal{F}, \mathbb{P})$, the probability space of the random environment. Hence, it is neither obvious what criteria to use nor how to proceed to prove convergence results for clock processes.

General criteria for the convergence clock processes were first introduced by Ben Arous and Černý in [17]. However, the formulation in [17] is restricted to the setting of random hopping dynamics, which refers to dynamics that are a time change of the simple random walk, i.e. processes X_n for which J_n is the simple random walk on G_n. In [17], conditions on the random environment and on the potential theory of J_n are given. Even though no particular correlation structure of the random environment is assumed, the techniques of [17] were only applied in models for which the random environment is a collection of i.i.d. random variables. The fact that the method put forward in [17] is not optimal in situations when the environment has a non-trivial correlation structure is reflected in [8], where the p-spin SK model is studied and results are only obtained in $\mathbb{P}$-law. A different and very general approach to study clock processes is proposed by Gayrard in [37]. There, abstract criteria for the convergence of sequences of partial sum processes are implemented in for clock processes of arbitrary Markov jump processes in random environments. The abstract criteria that are used in [37] are due to Durrett and Resnick, [33], and are designed to study partial sum processes that are constructed from arrays of correlated random variables. The power of the method introduced in [37] is illustrated in [37, 36], and [28], where applications to models are presented and further specializations are introduced. In particular, Bovier and Gayrard, [28], specialize the criteria of [37] to the setting when the random motion is defined on a sequence of finite graphs and when the increments of the clock process are highly correlated. This criterion is then used to improve the results of [8] to results that are valid $\mathbb{P}$-a.s., respectively in $\mathbb{P}$-probability. In this thesis, we further exploit the methods put forward in [37] and [28] and show how they can be extended to other limiting processes, respectively other graphs. Combining [37], [28], and the results of this thesis, one obtains a unified framework for proving convergence of clock processes. Let us explain this in more detail.

First let $\omega \in \Omega$ be fixed. Let us describe the abstract conditions that we use. In virtue of Theorem 1.6, we first introduce criteria for sequences of point processes to converge to Poisson random measures taken from [33]. We present the criteria from [33] in an abstract setting.

Theorem 1.9 (Theorem 3.1 in [33]). *Let $\{Z_{n,i} : i, n \geq 1\}$ be an array of positive random variables defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, a_n an increasing integer-valued sequence, and $\{\mathcal{F}_{n,i} : n, i \geq 1\}$ an array of sigma algebras such that $Z_{n,j}$ is $\mathcal{F}_{n,i}$ measurable for all $j \leq i$. Let ν be a sigma-finite measure on $(0,\infty)$ satisfying $\int_{(0,\infty)}(1 \vee x)\nu(dx) < \infty$. Suppose that for all $u > 0$ and $t > 0$ with $\nu(u,\infty) < \infty$ and $\nu(\{u\}) = 0$ the following two statements hold in $\mathcal{P}$-probability*

$$\nu_n^{Z,t}(u,\infty) \equiv \sum_{i=1}^{\lfloor a_n t \rfloor} \mathcal{P}(Z_{n,i} > u | \mathcal{F}_{n,i-1}) \to t\nu(u,\infty), \tag{1.3.1}$$

$$\sigma_n^{Z,t}(u,\infty) \equiv \sum_{i=1}^{\lfloor a_n t \rfloor} (\mathcal{P}(Z_{n,i} > u | \mathcal{F}_{n,i-1}))^2 \to 0. \tag{1.3.2}$$

Then, the point process $\xi_n \equiv \sum_{i \geq 1} \delta_{(i/a_n, Z_{n,i})}$ converges weakly in the space of point measures on $(0,\infty) \times (0,\infty)$ to a Poisson random measure ξ with intensity measure $dt \times d\nu$.

We use Theorem 1.9 to derive convergence criteria for sequences of partial sum processes, $\overline{S}_n(t) \equiv \sum_{i=1}^{a_n t} Z_{n,i}$, $t > 0$, as follows. Let $Z_{n,i}$ be an array of random variables, a_n an increasing sequence of integers, and ν a sigma finite measure for which the conditions of Theorem 1.9 hold. Denote by ξ the limiting Poisson random measure. We seek a continuous mapping so that with its help we can construct the partial sum process from ξ_n and a subordinator from ξ. In principle, such a map is given by the summation map T that maps a point process $m = \sum_{i \geq 1} \delta_{(t_i, x_i)}$ to the process $Tm(t)$, $t > 0$, where $Tm(t) = \sum_{t_i \leq t} x_i$. Applying this for $t > 0$ to ξ_n we get

$$T\xi_n(t) = \sum_{i=1}^{\lfloor a_n t \rfloor} Z_{n,i} = \overline{S}_n(t), \tag{1.3.3}$$

as desired. However, due to the large number of 'small' jumps, $T\xi(t)$, $t > 0$ is in general not always finite valued and so we cannot hope to get a convergence statement for $T\xi_n$ directly. Therefore, an additional condition is required. We use the same as in [37] which assumes that for all $t > 0$, we have

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \mathcal{E}\left[\sum_{i=1}^{\lfloor a_n t \rfloor} Z_{n,i} \mathbb{1}_{Z_{n,i} < \delta}\right] = 0, \tag{1.3.4}$$

where $\mathcal{E}$ denotes the expectation with respect to $\mathcal{P}$. Combining (1.3.4) and the conditions of Theorem 1.9, we obtain the following theorem.

Theorem 1.10 (Theorem 2.1 in [37]). *Suppose that* (1.3.4) *holds for all $t > 0$. Then, under the assumptions of Theorem 1.9, the partial sum process $\overline{S}_n$ converges to V_ν, which is a subordinator with Lévy measure ν and zero drift. Convergence holds weakly in the space $D[0,\infty)$ equipped with Skorohod's J_1 topology.*

Theorem 1.10 gives us abstract criteria for $\overline{S}_n$ to converge. In the following, we are also interested in slightly more general questions, namely convergence of $f_n(\overline{S}_n)$. The method we use in order to establish criteria for the convergence of $f_n(\overline{S}_n)$ is of the same spirit as that of Theorem 1.10. Namely, we first establish convergence of a sequence of point processes to a Poisson random measure and then apply a continuous mapping theorem to reconstruct $f_n(\overline{S}_n)$ from the sequence of point processes. With this in mind, one sees that the function f_n must be in such a way that, under some conditions, $f_n(\xi_n)$ is, in some sense, continuous. Furthermore, it is evident that processes which arise as limits of $f_n(\overline{S}_n)$ are of the form $f(\xi)$, where f is 'reasonably nice'. This plays a crucial rôle in Chapter 2 where we prove that $f_n(\overline{S}_n)$ converges for functions of the form $f_n(x) = x^{1/\alpha_n}$, where $\alpha_n \to 0$, to an extremal process. This rejoins the earlier mentioned fact that extremal processes arise as limits of α-stable subordinators when $\alpha \to 0$.

The theorems presented above state convergence criteria for an array of positive random variables $Z_{n,i}$ and we still must decide how to choose the $Z_{n,i}$ in the setting of clock processes. Also, due to the doubly stochastic nature of our processes, the verification of the conditions of Theorem 1.10, respectively Theorem 1.9, is in practice difficult. Therefore, we present in Chapter 2 and Chapter 3 new conditions, which are specialized to the setting of clock processes and imply those of Theorem 1.10, respectively Theorem 1.9. Let us describe this in more detail. The most natural choice, $Z_{n,i} = \lambda_n(J_n(i))e_{n,i}$, turns out to only apply to cases when the correlation between the random variables $Z_{n,i}$ is negligible. However, when the $Z_{n,i}$'s are correlated, either due to revisits of J_n or due to correlated random environments, it is known from [28] that one ought to proceed differently. Instead, one introduces a length $\theta_n \ll a_n$ and defines $Z_{n,i} \equiv \sum_{j=i\theta_n}^{(i+1)\theta_n} \lambda_n(J_n(i))e_{n,i}$. The idea behind this is to choose θ_n in such a way that the $Z_{n,i}$'s are essentially uncorrelated under $\mathcal{P}_{\mu_n}$. Doing so, we merge θ_n jumps of J_n into one jump of $f_n(S_n)$ and we loose information on the behavior of $f_n(S_n)$ which results in convergence in a weaker topology. Thus, one must be careful not to choose θ_n too large. How to choose θ_n is contained in a condition, which concerns only the distribution of J_n. This condition is constructed in such a way that the verification of Theorem 1.9 simplifies considerably. More precisely, we show that under the above mentioned assumptions on θ_n and J_n we may take the average over $\mathcal{P}_{\mu_n}$ in the $\nu_n^{t,Z}(u,\infty)$ and $\sigma_n^{t,Z}(u,\infty)$. In other words, in order to verify the conditions of Theorem 1.10 it then suffices to control quantities that are only random in the random environment. These are the types of criteria for the convergence of the 'delayed' clock process, i.e. for $f_n(S_n(t) - S_n(0))$, $t > 0$. To obtain convergence of the full clock process, $f_n(S_n)$, we evoke yet another condition, concerning the initial distribution μ_n which ensures that the initial jump, $S_n(0)$, does not contribute to the clock process. As can be seen in Chapter 2 and Chapter 3, these conditions vary with the functions f_n, respectively with the size of the graph (finite vs. infinite graphs).

In Chapter 4, we take another point of view and study the effect on correlation functions when the contribution coming from the initial distribution to the clock process is not negligible. The rôle of the initial distribution in the behavior of correlation functions was first made explicit in [37], and the only results in which the aging behavior changes due to the initial distribution are due to Gayrard in [38]. We follow a similar approach and establish in Chapter 4 show that, depending on the time scales on which we observe the process, the behavior of correlation functions might be completely governed by the initial jump, respectively initial block, of S_n.

The above mentioned criteria are stated for fixed $\omega \in \Omega$ and the question arises in which convergence mode with respect to $\mathbb{P}$ we seek statements. From [37] we know that the two possible convergence modes are $\mathbb{P}$-a.s. or in $\mathbb{P}$-probability: Suppose there exists a sequence $\Omega_n \subseteq \Omega$ with $\lim_{n\to\infty} \mathbb{P}(\Omega_n) = 1$, a sequence $\varepsilon_n \to 0$, and an increasing f such that for all $\omega \in \Omega_n$, for all $u > 0$ such that $\nu(\{u\}) = 0$ and for all $t > 0$,

$$|\nu_n^{Z,t}(u,\infty) - \nu(u,\infty)| < \varepsilon_n, \tag{1.3.5}$$

$$|\sigma_n^{Z,t}| < \varepsilon_n, \tag{1.3.6}$$

$$|\mathcal{E} \textstyle\sum_{i=1}^{\lfloor a_n t \rfloor} Z_{n,i} \mathbb{1}_{Z_{n,i}<\delta} - f(\delta)| < \varepsilon_n. \tag{1.3.7}$$

Then, if $\mathbb{P}(\cap_n \Omega_n) = 0$, the conditions of Theorem 1.9 and Theorem 1.10 are satisfied in $\mathbb{P}$-probability, or, if $\mathbb{P}(\cap_n \Omega_n) = 1$, they are satisfied $\mathbb{P}$-almost surely.

We conclude this section with a discussion of the main results of this thesis.

1.4 Main results of the thesis

Since the thesis consists of three independent papers, the description of the main results is divided into three distinct sections. The common ground between the three papers is that they develop universal mechanisms to study the aging behavior of Markov jump processes in random environments.

The results of the first and second paper are first presented in a general setting and then applied to various models. In the third part, we decided to study one particular model in depth even though the technique is universal. Let us explain this in more detail.

1.4.1 Convergence to extremal processes in random environments

The first part deals with dynamics defined on a sequence of finite graphs and establishes abstract criteria for the properly rescaled clock process to converge to an extremal process. This part, except for Sections 2.1.3 - 2.1.4 and Sections 2.3 - 2.4, has appeared in Probability Theory and Related Fields as the following joint work with Anton Bovier and Véronique Gayrard:

> A. Bovier, V. Gayrard, and A. Švejda. *Convergence to extremal processes in random environments and extremal ageing in SK models*, Probability Theory and Related Fields, Volume 157, Issue 1 (2013), pp. 251–283.

The main results of Chapter 2 can be divided into three parts. The first is a theorem stated in an abstract setting where we study sums, $S_n(k) = \sum_{i=1}^{k} Z_{n,i}$, of positive random variables, $Z_{n,i}$, by analogy to Theorem 1.10. It is known, [56, 33], that in order to get convergence to an α-stable subordinator, for $\alpha \in (0, 1)$, one requires the $Z_{n,i}$'s to have a regularly varying tail distribution with index $-\alpha$. The question we ask in Chapter 2 is what happens when α tends to zero? The same question is studied for subordinators in [44], where it is explained that 0-stable subordinators do not exist, but shown that by applying non-linear transformations to α_n-stable subordinators with $\alpha_n \to 0$, extremal processes arise as limit processes. Thus, in Theorem 2.1 we show how to choose sequences, a_n, α_n, satisfying $a_n \to \infty$ and $\alpha_n \to 0$ respectively, as $n \to \infty$, ensuring that the process $(S_n(\lfloor a_n t \rfloor))^{\alpha_n}$, $t \geq 0$, converges in a suitable sense to an extremal process. This theorem complements the results of [33] where it is shown that the maximum of a collection of random variables converges to an extremal process under suitable conditions. The second part consists of two theorems that specialize Theorem 2.1 for clock processes of Markov jump processes in random environments defined on sequences of infinite graphs (see Theorem 2.2 and Theorem 2.3). Theorem 2.2 translates Theorem 2.1 into this setting and Theorem 2.3 shows how the conditions of Theorem 2.1 simplify under the assumption that the jump chain J_n admits a unique invariant probability measure and is rapidly mixing.

Finally, we present in Sections 2.1.3-2.1.5 three applications of the above mentioned theorems. More precisely, we show in three models for which scales a_n, c_n, and α_n, the non-linearly rescaled clock process $f_n(S_n) = (S_n)^{\alpha_n}$, where S_n is as in (1.1.7), converges to an extremal process. Here, since convergence only holds for $f_n(S_n)$, the relations (1.1.4) or(1.1.5) can only be satisfied for $\mathcal{C}_n(t^{1/\alpha_n}, (t(1+\rho))^{1/\alpha_n} - t^{1/\alpha_n})$, and so we establish *extremal aging* in these models. In Sections 2.1.3 and 2.1.4 we study two models with i.i.d. random environments, namely the random hopping dynamics of Bouchaud's trap model on the complete graph (see Section 2.1.3) and the same dynamics of the REM (see Section 2.1.4). Even though the correlation structure of these random environments is trivial, it is important to understand their long time behavior because they belong to the same universality class with respect to their aging behavior; see [11, 13, 37, 8, 28] for normal aging and [42] for extremal aging. The results we obtain in Section 2.1.3 for a richer class of random environments than those in [42] and those in Section 2.1.4 for a wider range of time scales. Finally we apply our abstract results to a prominent example for models with non-trivial correlation structure - the p-spin SK models - in Section 2.1.5. The power of the abstract theorems, Theorem 2.2 and Theorem 2.3, is reflected in the fact that our results in Sections 2.1.3 - 2.1.5 improve all previously obtained results for convergence of the properly rescaled clock process to an extremal process (see [42] and [19]). It is important to remark that in particular Theorem 2.8, respectively Theorem 2.9, improve the results of [19] that hold in law with respect to the random environment, whereas our statements are true almost surely (for $p > 4$) respectively in probability (for $p = 2, 3, 4$).

1.4.2 Convergence of clock processes on infinite graphs

In the second part of this thesis we study dynamics defined on an infinite graph and establish abstract criteria for the properly rescaled clock process to converge to a subordinator. This part, except for Sections 3.1.4, 3.5, and 3.6.5, is available on the arxiv as the following joint work with Véronique Gayrard:

> V. Gayrard and A. Švejda. *Convergence of clock processes on infinite graphs and aging in Bouchaud's asymmetric trap model on* $\mathbb{Z}^d$, arxiv:1309.3066, 2013.

For $n \geq 1$ let $G_n = G_\infty$ be an infinite graph. For simplicity we write $G = (\mathcal{V}, \mathcal{L})$, X, J, and $\widetilde{S}$. In Chapter 3 we specialize the convergence criteria for sums of dependent random variables, Theorem 1.10, to the setting of Markov jump processes in random environments on infinite graphs. Two objects are of interest in this class of models depending on whether one is interested in the aging behavior or scaling limits of X. In the first case, one typically studies the clock process $\widetilde{S}$ as in (1.1.2), whereas in the second case, one typically studies a different clock process than $\widetilde{S}$, namely the continuous time clock process. Let us define the continuous time clock process. Let $\overline{J}$ be another Markov jump process with the same jump chain as X. Since X and $\overline{J}$ have the same jump chain, there exists a function $\overline{S}$ such that,

$$X(t) = J(\overline{S}^{\leftarrow}(t)), \tag{1.4.1}$$

where $\overline{S}^{\leftarrow}(t)$ denotes the generalized right-continuous inverse of $\overline{S}$. The function $\overline{S}$ is called the continuous time clock process and is given by

$$\overline{S}(t) \equiv \int_0^t \lambda^{-1}(J(s))\overline{\lambda}(J(s))ds, \qquad t \geq 0, \tag{1.4.2}$$

where $\overline{\lambda}(x)$ is the parameter of the exponential holding time of $\overline{J}$ in x. The aim of Chapter 3 is to formulate the criteria of [33], respectively Theorem 1.10, for $\widetilde{S}$ either being a discrete or a continuous time clock process on infinite graphs. Since the specialization to clock processes on finite graphs in [28] relies on the fact that J admits an invariant probability measure, which typically does not hold on infinite graphs, it is not obvious how to implement the criteria of [33] for clock processes on infinite graphs. In this respect it is important to notice that the reason why there exists no invariant probability measure for J is that the chain is either null-recurrent or transient. Therefore, one can hope to minimize correlation between the block variables of length θ_n by requiring that J does not revisit points too often. We establish in Theorem 3.1 that the invariant probability measure can, under the assumption that revisits along the subsequence $\{\theta_n i, 1 \leq i \leq a_n/\theta_n\}$ are negligible, be replaced by the empirical measure induced by $\{J(i\theta_n), 0 \leq i \leq a_n/\theta_n\}$, averaged over P_μ. We show that these assumptions are always satisfied for transient J's and cannot hold for positive recurrent J's. For null-recurrent J's they have to be verified by hand and yield criteria how to choose θ_n. In Chapter 3 we present a second theorem stated in a general setting. It is designed to simplify the conditions of Theorem 3.1 in the setting when J is random in the random environment, that is when the empirical measure averaged over P_μ is random in the environment. Specifically, in Theorem 3.3 we present an additional condition which holds whenever J satisfies a $\mathbb{P}$-a.s. uniform local central limit theorem (where uniform refers to space and time) and enables us to replace the empirical measure by a deterministic probability measure.

The third part of Chapter 3 consists of two applications of the abstract theorems (see Sections 3.1.3 and 3.1.4). In Section 3.1.3 we study Bouchaud's asymmetric trap model on $\mathbb{Z}^d$, $d \geq 2$ (see Section 1.4.3 below for a definition). This model is a prominent example for a dynamics that is random in the random environment and has so far been only studied in respect of its scaling limit. More precisely, it is shown in [53] ($d \geq 5$), [2] ($d \geq 3$) and [31] ($d = 2$) that a properly rescaled continuous time clock process converges to a subordinator. The continuous time clock process which is studied in alle these paper is obtained for the so-called *variable speed random walk*, $\widetilde{J}$, which is a central object in random conductance models and whose scaling limit is known (for the most recent results see [1], for the strongest results see [4], and for a general overview see [23]). Therefore, the scaling limit for X can be constructed using the convergence of $\widetilde{S}$ and the results on $\widetilde{J}$. In Section 3.1.3 we follow a similar approach and consider the same clock process and prove normal aging for classical, respectively natural, correlation functions. Then, in Section 3.1.4, we study a version of Bouchaud's trap model whose jump distribution is more general than that of the simple random walk but is deterministic in the random environment. We prove the existence of an $\mathbb{P}$-a.s. aging regime in this setting. This model has been studied previously in [34], where aging results are, for a richer class of models than in Section 3.1.4, obtained only in $\mathbb{P}$-law, respectively in $\mathbb{P}$-probability.

1.4.3 Super-aging in Bouchaud's asymmetric trap model on $\mathbb{Z}^d$

In the third part of this thesis we study the effect of initial distributions on the long time behavior of Bouchaud's asymmetric trap model on $\mathbb{Z}^d$ for $d \geq 3$ and its symmetric version for $d = 2$. It is based on joint work with Véronique Gayrard, which in turn in based on [38].

> V. Gayrard and A. Švejda. *Aging beyond the arcsine law and super-aging in Bouchaud's asymmetric trap model on $\mathbb{Z}^d$ (preliminary version).*

Let us first define the model that we study in Chapter 4. For $d \geq 2$ let $G = (\mathbb{Z}^d, \mathcal{L})$, where $\mathcal{L}$ is the collection of nearest-neighbor edges. The random environment, $\{\tau(x), x \in \mathbb{Z}^d\}$, is a collection of i.i.d. random variables, with tail distribution given by

$$\mathbb{P}(\tau(0) > u) = Cu^{-\alpha}(1 + L(u)), \quad u \geq \bar{c}, \tag{1.4.3}$$

where $\alpha \in (0, 1)$, $C, \bar{c} \in (0, \infty)$ are constants and $L : (0, \infty) \to \mathbb{R}$ tends to zero as $u \to \infty$. For $\theta \in [0, 1]$, the jump rates, λ, of Bouchaud's asymmetric trap model, X, are given by $\lambda(x, y) = (\tau(x))^{\theta-1}(\tau(y))^{\theta}$, if x and y are nearest neighbors, and zero else. Typically, the initial distribution, μ, of X is given by δ_0, where $0 \in \mathbb{Z}^d$ (and we establish in Chapter 3 that for $\mu = \delta_0$ the process exhibits normal aging). In Chapter 4 we ask the question under which conditions on μ this behavior is preserved, or more precisely, whether there exist initial distributions that change the aging behavior of the process. The correlation function that we study is the probability that $\{X(s) = X(s + t)\}$, denoted by $R(s, t)$. Similar questions were asked in [37] and [38]: in the former, a general limiting form for the correlation function defined in (1.1.6) is established and in the latter a class of initial distributions is introduced that slows down the dynamics so that correlation functions exhibit super-aging. The method used in [38] is universal, but the results are restricted to random hopping dynamics of mean-field models. In Chapter 4 we implement the techniques of [37] and [38] in the context of Bouchaud's asymmetric trap model on $\mathbb{Z}^d$, $d \geq 3$, and its symmetric version on $\mathbb{Z}^2$.

The first main result in Chapter 4 is Theorem 4.1 where we establish the limiting form of R for general initial distributions. Specifically, we introduce sufficient conditions on the time scale c_ℓ, the (possibly random) initial distribution μ, and the set $\overline{\mathcal{A}}_\ell = \{x : \mathbb{P}(\mu(x) > 0) > 0\}$ which contains all vertices in $\mathbb{Z}^d$ that have positive probability to belong to the support of μ. The first condition establishes the convergence of the initial block variable to a random variable σ having distribution function F on $[0, \infty)$. The second condition of Theorem 4.1 gives an upper bound the size of the support of μ, $|\mathcal{A}_\ell|$, as a function of the time scale c_ℓ. This upper bound can be explained in the following way. Since the τ's are in the domain of attraction of an α-stable law, X spends most of its time in sites x for which $\tau(x)$ is maximal, which is at most the order of $|\overline{\mathcal{A}}_\ell|^{1/\alpha}$. When X is transient, i.e. for $d \geq 3$, it spends in $\mathcal{A}_\ell$ at most a time of the order of $|\overline{\mathcal{A}}_\ell|^{1/\alpha}$, hence $|\overline{\mathcal{A}}_\ell|^{1/\alpha} \leq c_\ell$. When X is recurrent, i.e. for $d = 2$, it spends there on the time scale c_ℓ a time at most of the order of $|\overline{\mathcal{A}}_\ell|^{1/\alpha} \log c_\ell$, and therefore we ask that $\overline{\mathcal{A}}_\ell|^{1/\alpha} \leq c_\ell/(\log c_\ell)$. Under these assumption we establish in Theorem 4.1 that $R(c_\ell t_w, c_\ell t)$ converges, as $\ell \to \infty$, to

$$R_\infty(t_w, t) = 1 - F(t_w + t) + \textstyle\int_0^{t_w} \mathrm{Asl}_\alpha\big(\frac{t_w - v}{t_w + t - v}\big) dF(v), \tag{1.4.4}$$

where F is the distribution function of σ and Asl_α is the distribution function of the arcsine distribution. Note that if σ is a constant then (1.4.4) proves normal aging behavior of R. This is for example the case for all deterministic initial distributions having finite (but possibly diverging) support. We derive Theorem 4.1 from the convergence of the same continuous time clock process as in Chapter 3. Because the variable speed random walk $\widetilde{J}$ is typically studied for initial distributions $\mu = \delta_0$ we must adjust several results for $\widetilde{J}$ to the setting when the support of μ is increasing. The technique we use for this works in $d \geq 3$ and we therefore consider in $d = 2$ the symmetric version of this model (see Section 4.2.4 for explanations and further details).

We then apply Theorem 4.1 to the same class of initial distributions, $\mu_{A,b}$, as in [38]. Let us define these initial distributions. Let $A \subset \mathbb{Z}^d$ be such that $|A| \equiv \ell$, where ℓ is a diverging sequence. For $b > \alpha$, we start the process X according to the distribution

$$\mu_{A,b}(x) = \tfrac{\tau(x)^b}{\sum_{y \in A} \tau(y)^b}, \quad x \in A. \tag{1.4.5}$$

Using Theorem 4.1 we establish in Theorem 4.2 that, when we observe the process on time scales of the form $c_{\ell,\delta} = t_w^{-\delta} \ell^{1/\alpha}(\log \ell \mathbb{1}_{d=2} + \mathbb{1}_{d\geq 3})$, where $\delta \geq 1$, then $R(c_\ell t_w, c_\ell t)$ converges to $R_\infty(t_w, t)$ with F given by

$$F_{Y,\theta}(u) \equiv 1 - \textstyle\sum_{j=1}^\infty \frac{\gamma_j}{\sum_{j=1}^\infty \gamma_j} \exp\Big(-\rho/\big(t_w^\delta \gamma_j^{(1-\theta)/b} Y_i\big)\Big), \quad u \geq 0, \tag{1.4.6}$$

where $\Gamma = \sum_j \delta_{\gamma_j}$ is a Poisson random measure with intensity measure $\nu(u, \infty) = u^{-\alpha/b}$, $u > 0$, and $\{Y_j, j \in \mathbb{N}\}$ is a collection of i.i.d. random variables, independent of Γ. In Corollary 4.3 we prove that, for $\delta > 1$, $R_\infty(t_w, t)$ tends, as $t_w \to \infty$ and $t = t_w^\delta \rho$, to $1 - F_{Y,0}(\rho)$. Since $\delta > 1$, this proves the existence of a super-aging regime. This rejoins the earlier mentioned results for mean-field models in [38], where the correlation function is given by $1 - F_{Y,0}(\rho)$ for Y being deterministic. The initial distributions $\mu_{A,b}$ have the same form as the distribution of $X(t_w)$ in [58, 59]. Using this special form, the authors in [58, 59] discuss the possible occurrence of a sub-aging regime in Bouchaud's asymmetric trap model on $\mathbb{Z}^d$, $d \geq 1$, and find that the limiting correlation function is given by $1 - F_{Y,\theta}(\rho t_w^\delta)$. This prediction was only partially established for the one-dimensional model in [14]. Thus, Corollary 4.3 proves that the same correlation function arises in the description of the *super-aging* behavior in the higher dimensional models ($d \geq 2$).

Chapter 2

Convergence to extremal processes in random environments and extremal aging in SK models

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Abstract

We extend recent results on aging in mean field spin glasses on short time scales, obtained by Ben Arous and Gün [19] in law with respect to the environment, to results that hold almost surely, respectively in probability, with respect to the environment. It is based on the methods put forward in [37, 36] and naturally complements [28].

2.1 Introduction and main results

Spin glasses have, for the last decades, presented some of the most interesting challenges to probability theory. Even mean-field models have prompted a 1000 page monograph [61, 62] by one of the most eminent probablists of our time. Despite of these efforts and remarkable and unexpected progress, a full understanding of equilibrium problem, i.e. a full description of the asymptotic geometry of the Gibbs measures, is still outstanding. In this situation it is somewhat surprising that certain properties of their dynamics have been prone to rigorous analysis, at least for some limited choices of the dynamics. The reason for this is that interesting aspects of the dynamics occur on time-scales that are far shorter than the those of equilibration, and experiments made with spin glasses usually test the behavior of the probe on such time scales. Indeed, equilibration is expected to take so long as to become inaccessible to real experiments. The physically interesting issue is thus that of *aging* [24, 26], a property of time-time correlation functions that characterizes the slow decay to equilibrium characteristic for these systems.

The mathematical analysis has revealed an universal mechanism behind this phenomenon: the convergence of the *clock-process*, that relates the physical time to the number of “moves” of the process, to an α-stable subordinator (increasing Lévy process) under proper rescaling. The parameter α can be thought of as an *effective temperature* that depends both on the *physical temperature* and the *time scale* considered. This has been proven for p-spin Sherrington-Kirkpatrick (SK) models for time scales of the order $\exp(\beta\gamma n)$ (where n is the number of sites in the system with $0 < \gamma < \min(\beta, \zeta(p))$, where $\zeta(p)$ is an increasing function of p such that $\zeta(3) > 0$ and $\lim_{p\to\infty} \zeta(p) = 2\ln 2$. Such a result was obtained first in [8] *in law* with respect to the random environment, and was later extended in [28] to almost sure (resp. in probability, for $p = 3, 4$) results. The progress in the latter paper was possible to a fresh view on the convergence of clock processes, introduced and illustrated in two papers [37, 36] that takes the clock process as a sum of dependent random variables with a random distribution, and then employs conveniently suited convergence criteria, obtained by Durrett and Resnick [33] a long time ago, to prove convergence. This will be explained in more detail below.

The conditions on the admissible time scales in these results have two reasons. First, it emerges that $\alpha = \gamma/\beta$, so one of the conditions is simply that $\alpha \in (0, 1)$. The upper bound $\gamma < \zeta(p)$ ensures that there will be no strong long-distance correlations, meaning that the systems has not had time to discover the full correlation structure of the random environment. This condition is thus the stricter the smaller p is, since correlations become weaker as p increases.

A natural questions to ask is what happens on time-scales that are sub-exponential in the volume n? This question was first addressed in a recent paper by Ben Arous and Gün [19]. This situation would correspond formally to $\alpha = 0$, but 0-stable subordinators do not exist, so some new phenomenon has to appear. Indeed, they showed that the limiting objects appearing here are the so-called *extremal processes*. In the theory of sums of heavy tailed random variables this idea goes back to Kasahara [44] who showed that by applying non-linear transformations to the sums of α_n-stable r.v.'s with $\alpha_n \to 0$, extremal processes arise as limit processes. This program was implemented to clock processes by Ben Arous and Gün using the approach of [8] to handle the problems of dependence of the random variables involved. As a consequence, their results are again in law with respect to the random environment. An interesting aspect of this work was that, due to the very short time scales considered, the case $p = 2$, i.e. the original SK model, is also covered.

In the present paper we proceed along the line of [37, 36, 28] and formulate an abstract result which we apply to three different types of models, the most important one being the p-spin SK models. This way we show how one can extend the results of Ben Arous and Gün to *quenched* results, holding for given random environments almost surely (if $p > 4$) resp. in probability (if $2 \leq p \leq 4$).

Before stating our results, we begin by a concise description of the class of models we consider.

2.1.1 Markov jump processes in random environments

Let us describe the general setting of *Markov jump processes* in random environments that we consider here. Let $G_n(\mathcal{V}_n, \mathcal{L}_n)$ be a sequence of loop-free graphs with set of vertices $\mathcal{V}_n$ and set of edges $\mathcal{L}_n$. The *random environment* is a family of positive random variables, $\tau_n(x), x \in \mathcal{V}_n$, defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Note that in the most interesting situations the τ_n's are correlated random variables.

On $\mathcal{V}_n$ we consider a discrete time Markov chain J_n with initial distribution μ_n, transition probabilities $p_n(x, y)$, and transition graph $G_n(\mathcal{V}_n, \mathcal{L}_n)$. The law of J_n is a priori random on the probability space of the environment. We assume that J_n is reversible and admits a unique invariant measure π_n.

The process we are interested in, X_n, is defined as a time change of J_n. To this end we set

$$\lambda_n(x) \equiv C\pi_n(x)/\tau_n(x), \tag{2.1.1}$$

where $C > 0$ is a model dependent constant, and define the clock process

$$\widetilde{S}_n(k) = \sum_{i=0}^{k-1} \lambda_n^{-1}(J_n(i))e_{n,i}, \quad k \in \mathbb{N}\ , \tag{2.1.2}$$

where $\{e_{n,i} :\ i \in \mathbb{N}_0, n \in \mathbb{N}\}$ is an i.i.d. array of mean 1 exponential random variables, independent of J_n. The continuous time process X_n is then given by

$$X_n(t) = J_n(k), \quad \text{if } \widetilde{S}_n(k) \le t < \widetilde{S}_n(k+1) \quad \text{for some } k \in \mathbb{N},\ t > 0\ . \tag{2.1.3}$$

One verifies readily that X_n is a continuous time Markov jump process with infinitesimal generator

$$\lambda_n(x,y) \equiv \lambda_n(x)p_n(x,y), \tag{2.1.4}$$

and invariant measure that assigns to $x \in \mathcal{V}_n$ the mass $\tau_n(x)$.

To fix notation we denote by $\mathcal{F}^J$ and $\mathcal{F}^X$ the σ-algebras generated by the variables J_n and X_n, respectively. We write P_{π_n} for the law of the process J_n, conditional on $\mathcal{F}$, i.e. for fixed realizations of the random environment. Likewise we call $\mathcal{P}_{\mu_n}$ the law of X_n conditional on $\mathcal{F}$.

In [37, 36] and [28], the main aim was to find criteria when there are constants, a_n, c_n, satisfying $a_n, c_n \to \infty$, as $n \to \infty$, and such that the process

$$S_n(t) \equiv c_n^{-1}\widetilde{S}_n(\lfloor a_n t \rfloor) = c_n^{-1} \sum_{i=0}^{\lfloor a_n t \rfloor - 1} \lambda_n^{-1}(J_n(i))e_{n,i}, \quad t > 0, \tag{2.1.5}$$

converges in a suitable sense to a stable subordinator. The constants c_n are the time scale on which we observe the continuous time Markov process X_n, while a_n is the number of steps the jump chain J_n makes during that time. In order to get convergence to an α-stable subordinator, for $\alpha \in (0,1)$, one typically requires that the λ^{-1}'s observed on the time scales c_n have a regularly varying tail distribution with index $-\alpha$. In this paper we ask when there are constants, a_n, c_n, α_n, satisfying $a_n, c_n \to \infty$ and $\alpha_n \to 0$ respectively, as $n \to \infty$, and such that the process $(S_n)^{\alpha_n}$ converges in a suitable sense to an extremal process.

2.1.2 Main Theorems

We now state three theorems, beginning with an abstract one that we next specialize to the setting of Section 2.1.1. Specifically, consider a triangular array of positive random variables, $Z_{n,i}$, defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Let α_n and a_n be sequences such that $\alpha_n \to 0$ and $a_n \to \infty$ as $n \to \infty$, respectively. Our first theorem gives conditions that ensure that the sequence of processes $(S_n)^{\alpha_n}$, where $S_n(0) = 0$ and

$$S_n(t) \equiv \sum_{i=1}^{\lfloor a_n t \rfloor} Z_{n,i}, \quad t > 0, \tag{2.1.6}$$

converges to an extremal process.

Theorem 2.1. *Let ν be a sigma-finite measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ such that $\nu(0,\infty) = \infty$. Assume that there exist sequences a_n, α_n such that for all continuity points x of the distribution function of ν, for all $t > 0$, in $\mathcal{P}$-probability,*

$$\lim_{n\to\infty} \sum_{i=1}^{\lfloor a_n t \rfloor} \mathcal{P}\left(Z_{n,i}^{\alpha_n} > x | \mathcal{F}_{n,i-1}\right) = t\nu(x,\infty), \tag{2.1.7}$$

and

$$\lim_{n\to\infty} \sum_{i=1}^{\lfloor a_n t \rfloor} \left[\mathcal{P}\left(Z_{n,i}^{\alpha_n} > x | \mathcal{F}_{n,i-1}\right)\right]^2 = 0, \tag{2.1.8}$$

where $\mathcal{F}_{n,i}$ denotes the σ-algebra generated by the random variables $Z_{n,j}, j \le i$. If, moreover, for all $t > 0$

$$\limsup_{n\to\infty} \left(\sum_{i=1}^{\lfloor a_n t \rfloor} \mathcal{E}\mathbb{1}_{Z_{n,i} \le \delta^{1/\alpha_n}} \delta^{-1/\alpha_n} Z_{n,i} \right)^{\alpha_n} < \infty, \quad \forall\, \delta > 0\ , \tag{2.1.9}$$

then, as $n \to \infty$,

$$(S_n)^{\alpha_n} \overset{J_1}{\Longrightarrow} M_\nu, \tag{2.1.10}$$

where M_ν is an extremal process with one-dimensional distribution function $F(x) = e^{-\nu(x,\infty)}$. Convergence holds weakly on the space $D([0,\infty))$ equipped with the Skorokhod J_1-topology.

In the sequel we denote by $\overset{J_1}{\Longrightarrow}$ weak convergence in $D([0,\infty))$ equipped with the Skorokhod J_1-topology.

In order to use Theorem 2.1 in the Markov jump process setting of Section 2.1.1, we specify $Z_{n,i}$. In doing this we will be guided by the knowledge acquired in earlier works [37], [36], [28]: introducing a new scale θ_n we take $Z_{n,i}$ to be a block sum of length θ_n, i.e. we set

$$Z_{n,i} \equiv \sum_{j=(i-1)\theta_n+1}^{i\theta_n} c_n^{-1}\lambda_n^{-1}(J_n(j))e_{n,j} \,. \tag{2.1.11}$$

The rôle of θ_n is to de-correlate the variables $Z_{n,i}$ under the law $\mathcal{P}_{\mu_n}$. In models with uncorrelated environments and where the probability of revisiting points is small, one may hope to take $\theta_n = 1$. When the environment is correlated and the chain J_n is rapidly mixing, one may try to choose $\theta_n \ll a_n$ in such a way that, the variables $Z_{n,i}$ are close to independent. These two situations were encountered in the random hopping dynamics of the Random Energy Model in [36], and the p-spin models in [28] respectively. Theorem 2.2 below specializes Theorem 2.1 to these $Z_{n,i}$'s.

For $y \in \mathcal{V}_n$ and $u > 0$ let

$$Q_n^u(y) \equiv \mathcal{P}_y\left(\sum_{j=1}^{\theta_n} \lambda_n^{-1}(J_n(j))e_{n,j} > c_n u^{1/\alpha_n}\right) \tag{2.1.12}$$

be the tail distribution of the blocked jumps of X_n, when X_n starts in y. Furthermore, for $k_n(t) \equiv \lfloor \lfloor a_n t\rfloor/\theta_n \rfloor$, $t > 0$, and $u > 0$ define

$$\nu_n^{J,t}(u,\infty) \equiv \sum_{i=1}^{k_n(t)} \sum_{y\in\mathcal{V}_n} p_n(J_n(\theta_n i), y) Q_n^u(y) \,, \tag{2.1.13}$$

$$\sigma_n^{J,t}(u,\infty) \equiv \sum_{i=1}^{k_n(t)} \left[\sum_{y\in\mathcal{V}_n} p_n(J_n(\theta_n i), y) Q_n^u(y)\right]^2 \,. \tag{2.1.14}$$

Using this notation, we rewrite Conditions (2.1.7)-(2.1.9). Note that $Q_n^u(y)$ is a random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and so are the quantities $\nu_n^{J,t}(u,\infty)$ and $\sigma_n^{J,t}(u,\infty)$. The conditions below are stated for fixed realization of the random environment as well as for given sequences a_n, c_n, θ_n, and α_n such that $a_n, c_n \to \infty$, and $\alpha_n \to 0$ as $n \to \infty$.

Condition (1) Let ν be a σ-finite measure on $(0,\infty)$ with $\nu(0,\infty) = \infty$ and such that for all $t > 0$ and all $u > 0$

$$\lim_{n\to\infty} P_{\mu_n}\left(\left|\nu_n^{J,t}(u,\infty) - t\nu(u,\infty)\right| > \varepsilon\right) = 0 \,, \quad \forall \varepsilon > 0 \,. \tag{2.1.15}$$

Condition (2) For all $u > 0$ and all $t > 0$,

$$\lim_{n\to\infty} P_{\mu_n}\left(\sigma_n^{J,t}(u,\infty) > \varepsilon\right) = 0 \,, \quad \forall \varepsilon > 0 \,. \tag{2.1.16}$$

Condition (3) For all $t > 0$ and all $\delta > 0$

$$\limsup_{n\to\infty}\left(\sum_{i=1}^{\lfloor a_n t\rfloor} \mathcal{E}_{\mu_n} \mathbb{1}_{\{\lambda_n^{-1}(J_n(i))e_{n,i} \le \delta^{1/\alpha_n} c_n\}} (c_n \delta^{1/\alpha_n})^{-1} \lambda_n^{-1}(J_n(i))e_{n,i}\right)^{\alpha_n} < \infty \,. \tag{2.1.17}$$

Condition (0) For all $v > 0$,

$$\lim_{n\to\infty} \sum_{x\in\mathcal{V}_n} \mu_n(x) e^{-v^{1/\alpha_n} c_n \lambda_n(x)} = 0 \,. \tag{2.1.18}$$

For $t > 0$ set

$$\left(S_n^b(t)\right)^{\alpha_n} \equiv \left(\sum_{i=1}^{k_n(t)} \left(\sum_{j=\theta_n(i-1)+1}^{\theta_n i} c_n^{-1} \lambda_n^{-1}(J_n(j)) e_{n,j} \right) + c_n^{-1}\lambda_n^{-1}(J_n(0)) e_{n,0} \right)^{\alpha_n} . \tag{2.1.19}$$

Theorem 2.2. *If for a given initial distribution μ_n and given sequences a_n, c_n, θ_n, and α_n, Conditions (0)-(3) are satisfied $\mathbb{P}$-a.s., respectively in $\mathbb{P}$-probability, then*

$$\left(S_n^b\right)^{\alpha_n} \overset{J_1}{\Longrightarrow} M_\nu \,, \tag{2.1.20}$$

where convergence holds $\mathbb{P}$-a.s., respectively in $\mathbb{P}$-probability.

Remark. Theorem 2.2 tells us that the blocked clock process $(S_n^b)^{\alpha_n}$converges to M_ν weakly in $D([0,\infty))$ equipped with the Skorokhod J_1-topology. This implies that the clock process $(S_n)^{\alpha_n}$ converges to the same limit in the weaker M_1-topology (see [28] for further discussion).

Remark. The extra Condition (0) serves to guarantee that the last term in (2.1.19) is asymptotically negligible.

Finally, following [28], we specialize Conditions (1)-(3) under the assumption that the chain J_n obeys a mixing condition (see Condition (2-1) below). Conditions (1)-(2) of Theorem 2.2 are then reduced to laws of large numbers for the random variables $Q_n^u(y)$. Again we state these conditions for fixed realization of the random environment and given sequences a_n, c_n, θ_n, and α_n.

Condition (1-1) Let J_n be a periodic Markov chain with period q. There exists a positive decreasing sequence ρ_n, satisfying $\rho_n \to 0$ as $n \to \infty$, such that, for all pairs $x, y \in \mathcal{V}_n$, and all $i \geq 0$,

$$\sum_{k=0}^{q-1} P_{\pi_n}\left(J_n(i+\theta_n+k) = y, J_n(0) = x\right) \leq (1+\rho_n)\pi_n(x)\pi_n(y) \,. \tag{2.1.21}$$

Condition (2-1) There exists a σ-finite measure ν with $\nu(0,\infty) = \infty$ and such that

$$\nu_n^t(u,\infty) \equiv k_n(t) \sum_{x\in\mathcal{V}_n} \pi_n(x) Q_n^u(x) \to t\nu(u,\infty) \,, \tag{2.1.22}$$

and

$$\sigma_n^t(u,\infty) \equiv k_n(t) \sum_{x\in\mathcal{V}_n} \sum_{x'\in\mathcal{V}_n} \pi_n(x) p_n^{(2)}(x,x') Q_n^u(x) Q_n^u(x') \to 0 \,. \tag{2.1.23}$$

where $p_n^{(2)}(x,x') = \sum_{y\in\mathcal{V}_n} p_n(x,y) p_n(y,x')$ are the 2-step transition probabilities.

Condition (3-1) For all $t > 0$ and $\delta > 0$

$$\limsup_{n\to\infty} \left(\lfloor a_n t \rfloor \mathcal{E}_{\pi_n} \mathbb{1}_{\{\lambda_n^{-1}(J_n(1))e_{n,1} \leq c_n \delta^{1/\alpha_n}\}} c_n^{-1} \delta^{-1/\alpha_n} \lambda_n^{-1}(J_n(1)) e_{n,1} \right)^{\alpha_n} < \infty \,. \tag{2.1.24}$$

Theorem 2.3. *Let $\mu_n = \pi_n$. If for given sequences a_n, c_n, $\theta_n \ll a_n$, and α_n, Conditions (1-1)-(3-1) and (0) are satisfied $\mathbb{P}$-a.s., respectively in $\mathbb{P}$-probability, then $(S_n^b)^{\alpha_n} \overset{J_1}{\Longrightarrow} M_\nu$, $\mathbb{P}$-a.s., respectively in $\mathbb{P}$-probability.*

2.1.3 Application to Bouchaud's trap model on the complete graph

We apply Theorem 2.2 to Bouchaud's trap model on the complete graph, which is defined as follows. For $n \in \mathbb{N}$, the complete graph is given by $\mathcal{V}_n = \{1, \ldots, n\}$ and $\mathcal{L}_n = \{\{x, y\} : x, y \in \mathcal{V}_n\}$. We assign to each edge $x \in \mathcal{V}_n$ a random variable $\tau_n(x)$ and denote for $n \in \mathbb{N}$ the random environment by

$$\tau_n \equiv \{\tau_n(x) : x \in \mathcal{V}_n\}. \tag{2.1.25}$$

We assume that for every n, τ_n is a collection of independently and identically distributed random variables whose distribution function satisfies

$$\mathbb{P}(\tau_n(x) > u) = u^{-\alpha_n} L(u) \qquad u \in (0, \infty), \quad x \in \mathcal{V}_n, \tag{2.1.26}$$

where $(\alpha_n)_{n\in\mathbb{N}}$ is a given sequence such that $\alpha_n \in (0,1)$ for $n \in \mathbb{N}$, $\alpha_n \to 0$ as $n \to \infty$ and L is a slowly varying function, i.e.

$$\lim_{t\to\infty} \tfrac{L(tu)}{L(t)} = 1, \quad \forall u > 0. \tag{2.1.27}$$

We take J_n to be the simple random walk on $\mathcal{V}_n$, i.e. J_n has transition probabilities

$$p_n(x, y) = \frac{1}{n}, \qquad x, y \in \mathcal{V}_n. \tag{2.1.28}$$

This chain has unique invariant measure $\pi_n(x) = 1/n$ and we take $\mu_n = \pi_n$. We have the following convergence result for $S_n^{\alpha_n}$. Taking $C = 1/n$ in (2.1.1), the mean holding times, $\lambda_n^{-1}(x)$, reduce to $\lambda_n^{-1}(x) = \tau_n(x)$.

Theorem 2.4. *Let $(a_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ be increasing sequences such that*

(i) a_n satisfies in such a way that $a_n n^{-1} \log n \to 0$,

(ii) $a_n \mathbb{P}\left(\tau_n(1) > c_n u^{1/\alpha_n}\right) \to u^{-1}$ for every $u \in (0, \infty)$, $1 \in \mathcal{V}_n$,

(iii) there exists a constant $C > 0$, $N^\star \in \mathbb{N}$ such that for all $u \in (0, \infty)$ and $n \geq N^\star$

$$a_n\, \mathbb{P}(\tau_n(1) > c_n u) \leq C u^{-\alpha_n}. \tag{2.1.29}$$

Then we have, $\mathbb{P}$-a.s., ${S_n}^{\alpha_n} \overset{J_1}{\Longrightarrow} M$, where M is an extremal process with distribution function $F(x) = \exp(-1/x)$, $x \in (0, \infty)$.

We derive from Theorem 2.4 the limit of the correlation function $\mathcal{C}_n(s,t)$, which is for $t, s > 0$ given by

$$\mathcal{C}_n(t, s) = \mathcal{P}_{\mu_n}(X_n(c_n t^{1/\alpha_n}) = X_n(c_n v), \quad \forall t^{1/\alpha_n} \leq v \leq (t+s)^{1/\alpha_n}). \tag{2.1.30}$$

This is the probability that X_n did not jump during the time interval $(c_n t^{1/\alpha_n}, c_n(t+s)^{1/\alpha_n})$.

Theorem 2.5. *Under the assumptions of Theorem 2.4 we have that, $\mathbb{P}$-a.s.,*

$$\lim_{n\to\infty} \mathcal{C}_n(t, s) = \frac{t}{t+s}, \qquad t, s > 0. \tag{2.1.31}$$

Theorem 2.5 establishes extremal aging as defined in [19]. Here, de-correlation takes place on time intervals of the form $[t^{1/\alpha_n}, (t+s)^{1/\alpha_n}]$, while in normal aging it takes place on time intervals of the form $[t, t+s]$.

The results of Theorem 2.4 and Theorem 2.5 are established in [42] for the special case that $\tau_n(x) = \exp(1/\alpha_n e_n(x))$, where $\{e_n(x), x \in \mathcal{V}_n\}$ is a collection of i.i.d. mean one exponentially distributed random variables.

2.1.4 Application to the random energy model

In this section, we apply Theorem 2.2 to the Random Energy Model (hereafter referred to as REM). The underlying graph $\mathcal{V}_n$ is the hypercube $\Sigma_n = \{-1,1\}^n$ equipped with nearest neighbor edges. The Hamiltonian of the REM is a Gaussian process, H_n, on Σ_n with zero mean and covariance

$$\mathbb{E}H_n(x)H_n(x') = n\delta_{x=x'}. \tag{2.1.32}$$

The random environment, $\tau_n(x)$, is defined in terms of H_n through

$$\tau_n(x) \equiv \exp(\beta H_n(x)), \tag{2.1.33}$$

where $\beta \in \mathbb{R}_+$ is the inverse temperature. The Markov chain, J_n, is chosen as the simple random walk on Σ_n, i.e.

$$p_n(x,x') = \begin{cases} \frac{1}{n}, & \text{if } \operatorname{dist}(x,x') = 1, \\ 0, & \text{else,} \end{cases} \tag{2.1.34}$$

where the distance function on the hypercube is given by

$$\operatorname{dist}(x,x') \equiv \frac{1}{2}\sum_{i=1}^{n} |x_i - x_i'|. \tag{2.1.35}$$

This chain has unique invariant measure $\pi_n(x) = 2^{-n}$. Finally, choosing $C = 2^n$ in (2.1.1), the mean holding times, $\lambda_n^{-1}(x)$, reduce to $\lambda_n^{-1}(x) = \tau_n(x)$. This defines the so-called *random hopping dynamics*. We have the following convergence result for $S_n^{\alpha_n}$

Theorem 2.6. *Let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence satisfying*

(A) $\exists\eta > 0$ *such that* $\alpha_n \geq \left((7+\eta)\log n(n\beta^2)^{-1}\right)^{1/2}$, *and* $\alpha_n \to 0$.

Define $(a_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ via

(B) $a_n = \sqrt{2\pi n}\,\beta\alpha_n \exp\left(\frac{1}{2}\alpha_n^2 n\beta^2\right)$,

(C) $c_n = \exp\left(\alpha_n n\beta^2\right)$.

Then, $\mathbb{P}$-a.s., ${S_n}^{\alpha_n} \overset{J_1}{\Longrightarrow} M$, where M is an extremal process with distribution function $F(x) = \exp(-1/x)$ for $x > 0$.

We study the same correlation function as in Bouchaud's trap model on the complete graph, i.e. we take $\mathcal{C}_n$ as defined in (2.1.30) and obtain the following result.

Theorem 2.7. *Under the assumptions of Theorem 2.6 we have that, $\mathbb{P}$-a.s.,*

$$\lim_{n\to\infty} \mathcal{C}_n(t,s) = \frac{t}{t+s}, \qquad t,s > 0. \tag{2.1.36}$$

Our Theorem 2.6 and Theorem 2.7 are slight improvements of Theorem 48 in [42] because they admit a wider range of sequences α_n.

2.1.5 Application to the p-spin SK model.

In this section we illustrate the power of Theorem 2.3 by applying it to the p-spin SK models, including the SK model itself, i.e. $p \geq 2$. This model is a generalization of the REM, the only difference being in the definition of the Hamiltonian. The Hamiltonian of the p-spin SK model is a Gaussian process, H_n, on Σ_n with zero mean and covariance

$$\mathbb{E}H_n(x)H_n(x') = nR_n(x,x')^p, \tag{2.1.37}$$

where $R_n(x,x') \equiv 1 - \frac{2\,\mathrm{dist}(x,x')}{n}$. The random environment, $\tau_n(x)$, is defined in terms of H_n through

$$\tau_n(x) \equiv \exp(\beta H_n(x)), \tag{2.1.38}$$

where $\beta \in \mathbb{R}_+$ is again the inverse temperature. We take J_n to be the simple random walk on Σ_n, as defined in the previous section. Recall, that the unique invariant measure is given by $\pi_n(x) = 2^{-n}$. Finally, choosing $C = 2^n$ in (2.1.1), the mean holding times, $\lambda_n^{-1}(x)$, reduce to $\lambda_n^{-1}(x) = \tau_n(x)$. This defines the so-called *random hopping dynamics*.

In the theorem below the inverse temperature β is to be chosen as a sequence $(\beta_n)_{n\in\mathbb{N}}$ that either diverges or converges to a strictly positive limit.

Theorem 2.8. *Let ν be given by $\nu(u,\infty) \equiv K_p u^{-1}$ for $u \in (0,\infty)$ and $K_p = 2p$. Let γ_n, β_n be such that $\gamma_n = n^{-c}$ for $c \in \left(0, \frac{1}{2}\right)$, $\beta_n \geq \beta_0$ for some $\beta_0 > 0$, and $\gamma_n\beta_n \leq O(1)$. Set $\alpha_n \equiv \gamma_n/\beta_n$. Define the jump scales a_n and time scales c_n via*

$$a_n = \sqrt{2\pi n}\,\gamma_n^{-1}\,e^{\frac{1}{2}\gamma_n^2 n} \tag{2.1.39}$$
$$c_n = e^{\gamma_n\beta_n n}\,. \tag{2.1.40}$$

Then $\left(S_n^b\right)^{\alpha_n} \overset{J_1}{\Longrightarrow} M_\nu$. Convergence holds $\mathbb{P}$-a.s. for $p > 5$ and in $\mathbb{P}$-probability for $p = 2,3,4$. For $p = 5$ it holds $\mathbb{P}$-a.s. if $c \in \left(0, \frac{1}{4}\right)$ and in $\mathbb{P}$-probability else.

Remark. Theorem 2.8 immediately implies that $(S_n)^{\alpha_n} \overset{M_1}{\Longrightarrow} M_\nu$ on $D([0,\infty))$ equipped with the weaker M_1- topology.

In [19] an analogous result is proven in law with respect to the environment, for similar conditions on the sequence γ_n and fixed β.

Let us comment on the conditions on γ_n and β_n in Theorem 2.8. They guarantee that $\alpha_n \to 0$ as $n \to \infty$, and that both sequences a_n and c_n diverge as $n \to \infty$. Note here that different choices of the sequence β_n correspond to different time scales c_n. If $\beta_n \to \beta > 0$ as $n \to \infty$ then c_n is sub-exponential in n, while in the case of diverging β_n, c_n can be as large as exponential in $O(n)$. Finally these conditions guarantee that the rescaled tail distribution of the τ_n's, on time scale c_n, is regularly varying with index $-\alpha_n$.

We use Theorem 2.8 to derive the limiting behavior of the time correlation function $\mathcal{C}_n^\varepsilon(t,s)$ which, for $t > 0$, $s > 0$, and $\varepsilon \in (0,1)$ is given by

$$\mathcal{C}_n^\varepsilon(t,s) \equiv \mathcal{P}_{\pi_n}\left(A_n^\varepsilon(t,s)\right)\,, \tag{2.1.41}$$

where $A_n^\varepsilon(t,s) \equiv \left\{R_n\left(X_n(t^{1/\alpha_n}c_n), X_n((t+s)^{1/\alpha_n}c_n)\right) \geq 1-\varepsilon\right\}$.

Theorem 2.9. *Under the assumptions of Theorem 2.8,*

$$\lim_{n\to\infty} \mathcal{C}_n^\varepsilon(t,s) = \frac{t}{t+s}, \quad \forall\varepsilon \in (0,1),\ t,s > 0. \tag{2.1.42}$$

Convergence holds $\mathbb{P}$-a.s. for $p > 5$ and in $\mathbb{P}$-probability for $p = 2,3,4$. For $p = 5$ it holds $\mathbb{P}$-a.s. if $c \in \left(0, \frac{1}{4}\right)$ and in $\mathbb{P}$-probability else.

The remainder of the paper is organized as follows. We prove the results of Section 2.1.2 in Section 2.2. Section 2.5 is devoted to the proofs of the statements of Section 2.1.5. Finally, an additional lemma is proven in the Appendix.

2.2 Proofs of the main Theorems

Now we come to the proofs of the theorems of Section 2.1.2. The proof of Theorem 2.1 hinges on the property that extremal processes can be constructed from Poisson random measures. Namely, if $\xi' = \sum_{k\in\mathbb{N}} \delta_{\{t'_k,x'_k\}}$ is a Poisson random measures on $(0,\infty)\times(0,\infty)$ with intensity measure $dt \times d\nu'$, where ν' is a σ-finite measure such that $\nu'(0,\infty)=\infty$, then

$$M(t) \equiv \sup\{x'_k:\ t'_k \le t\}, \quad t>0, \tag{2.2.1}$$

is an extremal process with 1-dimensional marginal

$$F_t(u) = e^{-t\nu'(u,\infty)}. \tag{2.2.2}$$

(See e.g. [56], Chapter 4.3.). This was used in [33] to derive convergence of maxima of random variables to extremal processes from an underlying Poisson random measure convergence. Our proof exploits similar ideas and the key fact that the $1/\alpha_n$-norm converges to the sup norm as $\alpha_n \to 0$.

Proof of Theorem 2.1. Consider the sequence of point processes defined on $(0,\infty)\times(0,\infty)$ through

$$\xi_n \equiv \sum_{k\in\mathbb{N}} \delta_{\left\{k/a_n, Z_{n,k}^{\alpha_n}\right\}}. \tag{2.2.3}$$

By Theorem 3.1 of [33], Conditions (2.1.7) and (2.1.8) immediately imply that $\xi_n \overset{J_1}{\Longrightarrow} \xi$, where ξ is a Poisson random measure with intensity measure $dt\times d\nu$.

The remainder of the proof can be summarized as follows. In the first step we construct $(S_n(t))^{\alpha_n}$ from ξ_n by taking the α_n^{th} power of the sum over all points $Z_{n,k}$ up to time $\lfloor a_n t\rfloor$. To this end we introduce a truncation threshold δ and split the ordinates of ξ_n into

$$Z_{n,k}^{\alpha_n} = Z_{n,k}^{\alpha_n}\mathbb{1}_{Z_{n,k}^{\alpha_n}\le\delta} + Z_{n,k}^{\alpha_n}\mathbb{1}_{Z_{n,k}^{\alpha_n}>\delta}. \tag{2.2.4}$$

Applying a summation mapping to $Z_{n,k}^{\alpha_n}\mathbb{1}_{Z_{n,k}^{\alpha_n}>\delta}$, we show that the resulting process converges to the supremum mapping of a truncated version of ξ. More precisely, let $\delta>0$. Denote by $\mathcal{M}_p$ the space of point measures on $(0,\infty)\times(0,\infty)$. For $n\in\mathbb{N}$ let T_n^δ be the functional on $\mathcal{M}_p$, whose value at $m=\sum_{k\in\mathbb{N}}\delta_{\{t_k,j_k\}}$ is

$$(T_n^\delta m)(t) = \left(\sum_{t_k\le t} j_k^{1/\alpha_n}\mathbb{1}_{\{j_k>\delta\}}\right)^{\alpha_n}, \quad t>0. \tag{2.2.5}$$

Let T^δ be the functional on $\mathcal{M}_p$ given by

$$(T^\delta m)(t) = \sup\left\{j_k\mathbb{1}_{\{j_k>\delta\}} : t_k\le t\right\}, \quad t>0\,. \tag{2.2.6}$$

We show that $T_n^\delta\xi_n \overset{J_1}{\Longrightarrow} T^\delta\xi$ as $n\to\infty$.

In the second step we prove that the small terms, as $\delta\to0$ and $n\to\infty$, do not contribute to $(S_n)^{\alpha_n}$, i.e. that for $\varepsilon>0$

$$\lim_{\delta\to0}\limsup_{n\to\infty}\mathcal{P}\left(\rho_\infty\left(T_n^\delta\xi_n, S_n^{\alpha_n}\right)>\varepsilon\right) = 0\,, \tag{2.2.7}$$

where ρ_∞ denotes the Skorokhod metric on $D([0,\infty))$. Moreover, observe that $T^\delta\xi \overset{J_1}{\Longrightarrow} M$ as $\delta \to 0$. Then, by Theorem 4.2 from [22], the assertion of Theorem 2.1 follows.

Step 1*:* To prove that $T_n^\delta \xi_n \overset{J_1}{\Longrightarrow} T^\delta\xi$ as $n \to \infty$ we use a continuous mapping theorem, namely Theorem 5.5 from [22]. Since the mappings T_n^δ and T^δ are measurable, it is sufficient to show that the set

$$\mathcal{E} = \left\{ m \in \mathcal{M}_p : \exists\ (m_n)_{n\in\mathbb{N}} \text{ s.t. } m_n \overset{v}{\to} m \text{ but } T_n^\delta m_n \overset{J_1}{\not\Longrightarrow} T^\delta m \right\}, \tag{2.2.8}$$

where $\overset{v}{\to}$ denotes vague convergence in $\mathcal{M}_p$, is a null set with respect to the distribution of ξ. For the Poisson random measure ξ it is enough to show that $\mathcal{P}_\xi\left(\mathcal{E}^c \cap \mathcal{D}\right) = 1$, where

$$\mathcal{D} \equiv \left\{ m \in \mathcal{M}_p : m\left((0,t] \times [j,\infty)\right) < \infty\ \forall t, j > 0 \right\}. \tag{2.2.9}$$

Let $\mathcal{C}_{T^\delta} \equiv \left\{ t > 0 :\ \mathcal{P}_\xi\left(\left\{ m :\ T^\delta m\,(t) = T^\delta m\,(t-) \right\}\right) = 1 \right\}$ be the set of continuity points of ξ. By definition of the Skorokhod metric, we consider $m \in \mathcal{D}$, $a, b \in \mathcal{C}_{T^\delta}$, and $(m_n)_{n\in\mathbb{N}}$ such that $m_n \overset{v}{\to} m$ and show that

$$\lim_{n\to\infty} \rho_{[a,b]}\left(T_n^\delta m_n, T^\delta m\right) = 0\,, \tag{2.2.10}$$

where $\rho_{[a,b]}$ denotes the Skorokhod metric on $[a,b]$. Since $m \in \mathcal{D}$, there exist continuity points x, y of m such that $m((a,b) \times (\delta,\infty)) = m((a,b) \times (x,y)) < \infty$. Then, Lemma 2.1 from [52] yields that m_n also has this property for large enough n. Moreover, the points of m_n in $(a,b) \times (x,y)$ converge to the ones of m (cf. Lemma I.14 in [54]). Finally, we use that $\alpha_n \to 0$ as $n \to \infty$ and thus T_n^δ can be viewed as the $1/\alpha_n$-norm, which converges as $n \to \infty$ to the sup-norm T^δ. Therefore, $T_n^\delta \xi_n \overset{J_1}{\Longrightarrow} T^\delta \xi$ as $n \to \infty$.

Step 2*:* We prove (2.2.7) by showing that the assertion holds true for the Skorokhod metric on $D([0,k])$ for every $k \in \mathbb{N}$. Assume without loss of generality that $k = 1$. Let $\varepsilon > 0$. We have that

$$\begin{aligned} \mathcal{P} &\left(\sup_{0\le t\le 1}\left|T_n^\delta \xi_n\,(t) - S_n^{\alpha_n}\,(t)\right| > \varepsilon\right) \\ = &\ \mathcal{P}\left(\sup_{0\le t\le 1}\left|\left(\sum_{i=1}^{\lfloor a_n t\rfloor} Z_{n,i}\mathbb{1}_{Z_{n,i}>\delta^{1/\alpha_n}}\right)^{\alpha_n} - \left(\sum_{i=1}^{\lfloor a_n t\rfloor} Z_{n,i}\right)^{\alpha_n}\right| > \varepsilon\right). \end{aligned} \tag{2.2.11}$$

Since for n large enough $\alpha_n < 1$, we know by Jensen's inequality that

$$\left|\left(\sum_{i=1}^{\lfloor a_n t\rfloor} Z_{n,i}\mathbb{1}_{Z_{n,i}>\delta^{1/\alpha_n}}\right)^{\alpha_n} - \left(\sum_{i=1}^{\lfloor a_n t\rfloor} Z_{n,i}\right)^{\alpha_n}\right| \le \left|\sum_{i=1}^{\lfloor a_n t\rfloor} Z_{n,i}\mathbb{1}_{Z_{n,i}\le\delta^{1/\alpha_n}}\right|^{\alpha_n}, \tag{2.2.12}$$

and therefore

$$(2.2.11) \le \mathcal{P}\left(\sup_{0\le t\le 1}\left|\sum_{i=1}^{\lfloor a_n t\rfloor} Z_{n,i}\mathbb{1}_{Z_{n,i}\le\delta^{1/\alpha_n}}\right|^{\alpha_n} > \varepsilon\right)\,. \tag{2.2.13}$$

All summands are non-negative. Hence the supremum is attained for $t = 1$. Applying a first order Chebychev and Jensen's inequality, we obtain that (2.2.13) is bounded above by

$$\varepsilon^{-1}\left(\sum_{i=1}^{a_n} \mathcal{E}\mathbb{1}_{Z_{n,i}\le\delta^{1/\alpha_n}} Z_{n,i}\right)^{\alpha_n} = \frac{\delta}{\varepsilon}\left(\sum_{i=1}^{a_n} \mathcal{E}\mathbb{1}_{Z_{n,i}\le\delta^{1/\alpha_n}} \delta^{-1/\alpha_n} Z_{n,i}\right)^{\alpha_n}\,. \tag{2.2.14}$$

By (2.1.9) the sum is bounded in n and hence, as $\delta \to 0$, (2.2.14) tends to zero. This concludes the proof of Theorem 2.1. □

Proof of Theorem 2.2. Throughout we fix a realisation $\omega \in \Omega$ of the random environment but do not make this explicit in the notation. We set

$$\widehat{S}_n^b(t) \equiv S_n^b(t) - c_n^{-1}\lambda_n^{-1}(J_n(0))e_{n,0}, \quad t > 0. \tag{2.2.15}$$

$(S_n^b(t))^{\alpha_n}$ differs from $(\widehat{S}_n^b(t))^{\alpha_n}$ by one term. All terms in $(S_n^b(t))^{\alpha_n}$ are non-negative and therefore we conclude by Jensen's inequality that, for n large enough,

$$\widehat{S}_n^b(t)^{\alpha_n} \le S_n^b(t)^{\alpha_n} \le \widehat{S}_n^b(t)^{\alpha_n} + \left(c_n^{-1}\lambda_n^{-1}(J_n(0))e_{n,0}\right)^{\alpha_n}\,. \tag{2.2.16}$$

By Condition (0) the contribution of the term $\left(c_n^{-1}\lambda_n^{-1}(J_n(0))e_{n,0}\right)^{\alpha_n}$ is negligible. Thus we must show that under Conditions (1)-(3), $(\widehat{S}_n^b)^{\alpha_n} \xRightarrow{J_1} M_\nu$. Recall that $k_n(t) \equiv \lfloor\lfloor a_n t\rfloor/\theta_n\rfloor$ and that for $i \geq 1$,

$$Z_{n,i} \equiv \sum_{j=\theta_n(i-1)+1}^{\theta_n i} c_n^{-1}\lambda_n^{-1}(J_n(j))e_{n,j}. \tag{2.2.17}$$

We apply Theorem 2.1 to the $Z_{n,i}$'s. It is shown in the proof of Theorem 1.2 in [28] that Conditions (1) and (2) imply (2.1.7) and (2.1.8). It remains to prove that Condition (3) yields (2.1.9). Note that for all $i \geq 1$ and all $(i-1)\theta_n + 1 \leq j \leq i\theta_n$

$$\mathbb{1}_{\{\sum_{j=(i-1)\theta_n+1}^{i\theta_n} \lambda_n^{-1}(J_n(j))e_{n,j} \leq c_n\delta^{1/\alpha_n}\}} \leq \mathbb{1}_{\{\lambda_n^{-1}(J_n(j))e_{n,j} \leq c_n\delta^{1/\alpha_n}\}} . \tag{2.2.18}$$

Using (2.2.18), we observe that (2.1.9) is in particular satisfied if for every $\delta > 0$ and $t > 0$

$$\limsup_{n\to\infty}\Big(\sum_{i=1}^{\lfloor a_n t\rfloor} \mathcal{E}_{\mu_n}\mathbb{1}_{\{\lambda_n^{-1}(J_n(j))e_{n,j}\leq c_n\delta^{1/\alpha_n}\}}\delta^{-1/\alpha_n}c_n^{-1}\lambda_n^{-1}(J_n(j))e_{n,j}\Big)^{\alpha_n} < \infty\,, \tag{2.2.19}$$

which is nothing but Condition (3). The proof of Theorem 2.2 is done. □

Finally, having Theorem 2.2 and the results from [28], Theorem 2.3 is deduced readily.

Proof of Theorem 2.3. Let μ_n be the invariant measure π_n of the jump chain J_n. By Proposition 2.1 of [28] we know that Conditions (0), (1-1), and (2-1) imply Conditions (0)-(2) of Theorem 2.2. Moreover, since $\mu_n = \pi_n$, Condition (3-1) is Condition (3). Thus, the conditions of Theorem 2.2 are satisfied under the assumptions of Theorem 2.3 and this yields the claim. □

2.3 Application to Bouchaud's trap model on the complete graph

We show that the conditions of Theorem 2.2 are satisfied, $\mathbb{P}$-a.s., for the choice of a_n, c_n, and α_n of Theorem 2.4 and for the block length $\theta_n \equiv 1$. We begin with the verification of Condition (1). Let $u, t > 0$. In this setting, $\nu_n^{J,t}(u,\infty)$ is given by

$$\begin{aligned}\nu_n^{J,t}(u,\infty) &= \sum_{k=1}^{k_n(t)}\sum_{y\in\mathcal{V}_n} p_n(J(k),y)Q_n^u(y)\\ &= 1/n\sum_{k=1}^{k_n(t)}\sum_{y\in\mathcal{V}_n} Q_n^u(y) = \lfloor a_n t\rfloor/n\sum_{y\in\mathcal{V}_n} Q_n^u(y)\end{aligned} \tag{2.3.1}$$

To shorten notation, we set $\gamma_n(x) = c_n^{-1}\tau_n(x)$ for $x \in \mathcal{V}_n$. For $n \in \mathbb{N}$ and $y \in \mathcal{V}_n$ we note that, by the assumptions on J_n and the $e_{n,k}$'s

$$\begin{aligned}Q_n^u(y) &= \mathcal{P}_y\left(\gamma_n\left(J_n\left(k\right)\right)e_{n,k} > u^{1/\alpha_n}\right)\\ &= \sum_{i=1}^n \mathcal{P}_y\left(\gamma_n(i)e_{n,k} > u^{1/\alpha_n}, J_n\left(k\right) = i\right)\\ &= \sum_{i=1}^n \mathcal{P}_y\left(J_n\left(k\right) = i\right)\mathcal{P}_y\left(\gamma_n(i)e_{n,k} > u^{1/\alpha_n}\right)\\ &= 1/n\sum_{i=1}^n \exp\left(-u^{1/\alpha_n}/\gamma_n(i)\right).\end{aligned} \tag{2.3.2}$$

Together with (2.3.1), we find that

$$\nu_n^{J,t}(u,\infty) = \left(k_n(t)/a_n\right)a_n/n\sum_{i=1}^n \exp\left(-u^{1/\alpha_n}/\gamma_n(i)\right). \tag{2.3.3}$$

Note that this is not a random variable with respect to the chain, respectively the exponentials. Since moreover $k_n(t)/a_n \to t$ as $n \to \infty$, it suffices to show that there exists Ω_0 with $\mathbb{P}\left(\Omega_0\right) = 1$ such that on Ω_0 we have

$$\lim_{n\to\infty} a_n/n\sum_{i=1}^n \exp\left(-u^{1/\alpha_n}/\gamma_n(i)\right) = u^{-1}. \tag{2.3.4}$$

To this end we use the following strategy. We first show that

$$\lim_{n\to\infty} a_n\, \mathbb{E}\left[\exp\left(-u^{1/\alpha_n}/\gamma_n(1)\right)\right] = u^{-1}. \tag{2.3.5}$$

Then, we prove that there exists Ω_0 with $\mathbb{P}(\Omega_0) = 1$ such that on Ω_0

$$\lim_{n\to\infty} a_n \left|\frac{1}{n}\sum_{i=1}^{n} \exp\left(-u^{1/\alpha_n}/\gamma_n(i)\right) - \mathbb{E}\left[\exp\left(-u^{1/\alpha_n}/\gamma_n(1)\right)\right]\right| = 0. \tag{2.3.6}$$

Denote by $f_{\tau_n(1)}$ the density function of $\tau_n(1)$. By partial integration and substitution we find that

$$\begin{aligned} a_n\, \mathbb{E}\left[\exp\left(-u^{1/\alpha_n}\gamma_n(1)\right)\right] &= a_n \int_0^\infty dt \exp\left(-c_n u^{1/\alpha_n}/t\right) f_{\tau_n(1)}(t) \\ &= a_n \int_0^\infty dt \exp\left(-c_n u^{1/\alpha_n}/t\right)\ c_n u^{1/\alpha_n} t^{-2}\mathbb{P}(\tau_n(1) > t) \\ &= a_n \int_0^\infty dy\, e^{-y}\, \mathbb{P}\left(\gamma_n(1) > u^{1/\alpha_n}/y\right). \end{aligned} \tag{2.3.7}$$

By (2.1.26) and assumption (ii) we have that

$$\lim_{n\to\infty} \left|a_n e^{-y}\, \mathbb{P}\left(\gamma_n(1) > u^{1/\alpha_n}/y\right) - e^{-y}\, y^{\alpha_n}\, u^{-1}\right| = 0. \tag{2.3.8}$$

and moreover, by assumption (iii) we have for $n \geq N^\star$

$$a_n e^{-y}\, \mathbb{P}\left(\gamma_n(1) > u^{1/\alpha_n}/y\right) \leq C e^{-y}\, y^{\alpha_n}\, u^{-1} \leq C u^{-1} e^{-y}\ \left(\mathbb{1}_{\{y<1\}} + y\mathbb{1}_{\{1\leq y\}}\right). \tag{2.3.9}$$

Therefore, by the dominated convergence theorem, as $n \to \infty$,

$$(2.3.7) \sim \int_0^\infty dy e^{-y}\, y^{\alpha_n}\, u^{-1} = u^{-1}\,\Gamma(1+\alpha_n) \overset{n\to\infty}{\longrightarrow} u^{-1}.$$

This shows that (2.3.5) is satisfied. Let us now establish that (2.3.6) holds $\mathbb{P}$-a.s. To this end, we use the following concentration inequality, whose proof can be found in [20].

Proposition 2.10 (Bennett's inequality, [20]). *Let I be a finite set and $\{Y(k), k \in I\}$ a collection of independent random variables defined on some probability space $\left(\widetilde{\Omega}, \widetilde{\mathcal{F}}, P\right)$ with $E[Y(k)] = 0$ for every $k \in I$. Let $a \geq \max_{k\in I} ||Y_k||_\infty$ and $b^2 = \sum_{k\in I} E\left[Y_k^2\right]$. Then, for any $t > 0$, $\bar{b}^2 \geq b^2$*

$$\mathbb{P}\left(\sum_{k\in I} Y_k > t\right) \leq \exp\left(t/a - \left(t/a + \bar{b}^2/a^2\right)\log\left(1 + at/\bar{b}^2\right)\right). \tag{2.3.10}$$

Moreover, if $t \leq \bar{b}^2/(2a)$ the inequality simplifies to

$$\mathbb{P}\left(\sum_{k\in I} Y_k > t\right) \leq \exp\left(-t^2/(4\bar{b}^2)\right). \tag{2.3.11}$$

We apply (2.3.11) of Proposition 2.10 to the random variables

$$Y_n(k) = \exp\left(-u^{1/\alpha_n}/\gamma_n(k)\right) - \mathbb{E}\left[\exp\left(-u^{1/\alpha_n}/\gamma_n(1)\right)\right], \quad k = 1, \ldots, n. \tag{2.3.12}$$

Let $\varepsilon > 0$, $a = 2$ and $t = n/a_n\varepsilon$. Note for the choice of $\bar{b}^2$ that

$$\sum_{k=1}^{n} \mathbb{E}\left[Y_n^2(k)\right] = n\mathbb{V}\mathrm{ar}\left(\exp\left(-c_n u^{1/\alpha_n}/\gamma_n(1)\right)\right) \leq n\mathbb{E}\left[\exp\left(-2u^{1/\alpha_n}/\gamma_n(1)\right)\right]. \tag{2.3.13}$$

By similar calculations as in (2.1.25) and by assumptions (ii) and (iii) we find that

$$\lim_{n\to\infty} \left|a_n\, \mathbb{E}\left[\exp\left(-2u^{1/\alpha_n}/\gamma_n(1)\right)\right] - 2^{-\alpha_n}\Gamma(1+\alpha_n)\, u^{-1}\right| = 0. \tag{2.3.14}$$

Therefore, we have for n large enough

$$a_n\,\mathbb{E}\left[\exp\left(-2u^{1/\alpha_n}/\gamma_n(1)\right)\right] \le 2^{-\alpha_n}\Gamma(1+\alpha_n)\,u^{-1}+\varepsilon\,, \tag{2.3.15}$$

and hence we choose $\bar{b}^2 = n/a_n\left(2^{-\alpha_n}\Gamma(1+\alpha_n)\,u^{-1}+\varepsilon\right)$. Finally, we have to check whether $t \le \frac{b^2}{2a}$. This is satisfied if and only if

$$\varepsilon \le 4/32^{-\alpha_n}\Gamma(1+\alpha_n)\,u^{-1}\,, \tag{2.3.16}$$

which for n large enough is satisfied because the right hand side of (2.3.16) is increasing in n. Thus we find, for n large enough that

$$\begin{aligned}&\mathbb{P}\left(a_n\left|1/n\textstyle\sum_{k=1}^{n}\exp\left(-u^{1/\alpha_n}/\gamma_n(k)\right)-\mathbb{E}\left[\exp\left(-u^{1/\alpha_n}/\gamma_n(1)\right)\right]\right|>\varepsilon\right)\\ &\le 2\exp\left(-\varepsilon^2 2^{\alpha_n}un(a_n\Gamma(1+\alpha_n))^{-1}\right)\\ &\le\ 2\exp(-\varepsilon^2/4un/a_n),\end{aligned} \tag{2.3.17}$$

where we used the fact that $2^{\alpha_n}/\Gamma(1+\alpha_n)\ge 1$ for n large enough. By assumption (i) we have $n/a_n = n^{\kappa_n}$ where $\kappa_n\in(0,1)$ for all n. Hence we conclude by Borel Cantelli Lemma that there exists Ω_0 with $\mathbb{P}(\Omega_0)=1$ such that (2.3.6) holds on Ω_0. This finishes the verification of Condition (1).

We show now that Condition (2) is satisfied $\mathbb{P}$-a.s. for $u,t>0$. We rewrite $\sigma_n^{J,t}$, using (2.3.2) and (2.3.1) to find that

$$\begin{aligned}\sigma_n^{J,t}(u,\infty) &= \textstyle\sum_{k=1}^{\lfloor a_n t\rfloor}\left(\sum_{y\in\mathcal{V}_n}p_n(J(k),y)Q_n^u(y)\right)^2\\ &= \lfloor a_n t\rfloor/n^2\left(\textstyle\sum_{y\in\mathcal{V}_n}Q_n^u(y)\right)^2\\ &\le\ (k_n(t))^{-1}\left(\nu_n^{J,t}(u,\infty)\right)^2.\end{aligned} \tag{2.3.18}$$

Note that this is again only a random variable in the random environment. Let $\varepsilon>0$. By (2.3.5) and (2.3.6) we know that there exists Ω_0 with $\mathbb{P}(\Omega_0)=1$ such that on Ω_0 we have for n large enough that

$$\nu_n^{J,t}(u,\infty)\le u^{-1}(1+\varepsilon). \tag{2.3.19}$$

Using this in (2.3.18), we see that, $\mathbb{P}$-a.s., for n large enough

$$\sigma_n^{J,t}(u,\infty)\le(k_n(t))^{-1}u^{-2}(1+\varepsilon)^2, \tag{2.3.20}$$

which tends to zero as $n\to\infty$ and we deduce that Condition (2) is satisfied $\mathbb{P}$-a.s. The verification of Condition (0) follows similarly by noting that

$$\textstyle\sum_{y\in\mathcal{V}_n}\pi_n(y)\exp(-c_n v^{1/\alpha_n}/\tau_n(y)) = a_n^{-1}\nu_n^{J,1}(v,\infty). \tag{2.3.21}$$

It remains to establish that Condition (3) is satisfied $\mathbb{P}$-a.s. That is, we show that, $\mathbb{P}$-a.s.,

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\left(\delta^{-1/\alpha_n}c_n^{-1}\mathcal{E}_{\pi_n}\sum_{k=1}^{a_n}\tau_n(J_n(k))e_{n,k}\mathbb{1}_{\tau_n(J_n(k))e_{n,k}\le c_n\delta^{1/\alpha_n}}\right)^{\alpha_n}<\infty \tag{2.3.22}$$

The expected value in (2.3.22) is by a substitution given by

$$\begin{aligned}&\delta^{-1/\alpha_n}\mathcal{E}_{\pi_n}\left[\textstyle\sum_{k=1}^{a_n}\gamma_n(J_n(k))e_{n,k}\mathbb{1}_{\gamma_n(J_n(k))e_{n,k}\le\delta^{1/\alpha_n}}\right]\\ &= \delta^{-1/\alpha_n}c_n^{-1}\textstyle\sum_{k=1}^{a_n}\int_0^{c_n\delta^{1/\alpha_n}}\mathcal{P}_{\pi_n}\left(\tau_n\left(J_n(k)\right)e_{n,k}>x\right)dx\\ &= \textstyle\sum_{k=1}^{a_n}\int_0^1 dv\,\mathcal{P}_{\pi_n}\left(\gamma_n\left(J_n(k)\right)e_{n,k}>v\delta^{1/\alpha_n}\right).\end{aligned} \tag{2.3.23}$$

Since we are taking the α_n^{th} power in (2.3.22) and since $\alpha_n \to 0$, it is sufficient to bound (2.3.23) by a term that is polynomial in α_n. Moreover, since we are taking first $n \to \infty$ and then $\delta \to 0$, this bound may depend on δ. By the definition of J_n and the $e_{n,k}$'s, the sum in (2.3.23) equals

$$\begin{aligned}\sum_{k=1}^{a_n} \int_0^1 dv \mathcal{P}_{\pi_n}\left(\gamma_n\left(J_n\left(k\right)\right) e_{n,k} > v\delta^{1/\alpha_n}\right) &= \frac{a_n}{n} \int_0^1 dv \sum_{i=1}^n \mathcal{P}_{\pi_n}\left(\gamma_n\left(i\right) e_{n,1} > v\delta^{1/\alpha_n}\right) \\ &= \frac{a_n}{n} \int_0^1 dv \sum_{i=1}^n \exp\left(-\frac{\delta^{1/\alpha_n} v}{\gamma_n(i)}\right). \qquad (2.3.24)\end{aligned}$$

As in the verification of Condition (1) now use a law of large numbers for the terms depending on τ_n. Setting $F_n\left(v\right) = 1/n \sum_{i=1}^n \exp\left(-\delta^{1/\alpha_n} v/\gamma_n\left(k\right)\right)$, we find that

$$(2.3.24) = a_n \int_0^1 dv F_n\left(v\right) = a_n \int_0^1 dv \left(F_n\left(v\right) - \mathbb{E}\left[F_n\left(v\right)\right]\right) + a_n \int_0^1 dv \mathbb{E}\left[F_n(v)\right]. \quad (2.3.25)$$

Let $v_k \equiv k/n$. Since F_n is decreasing in v, the integral of the function $F_n - \mathbb{E}F_n$ in (2.3.25) is bounded above by

$$\begin{aligned}(2.3.25) \quad &\le \int_0^{v_1} dv a_n + a_n \sum_{k=1}^{n-1} \int_{v_k}^{v_{k+1}} dv \left(F_n\left(v\right) - \mathbb{E}\left[F_n\left(v\right)\right]\right) + a_n \int_0^1 dv \mathbb{E}\left[F_n\left(v\right)\right] \\ &\le a_n/n + a_n \sum_{k=1}^{n-1} dv \left(F_n\left(v_k\right) - \mathbb{E}\left[F_n\left(v_{k+1}\right)\right]\right) 1/n + a_n \int_0^1 dv \mathbb{E}\left[F_n\left(v\right)\right] \\ &\le 1 + (I) + (II), \qquad (2.3.26)\end{aligned}$$

where we used assumption (i) for the bound $a_n n^{-1} \le 1$. We first bound (II). For $n \ge N^\star$ we have that

$$\begin{aligned}(II) &= \int_0^1 dv\, a_n \int_0^\infty dt \mathbb{P}\left(\gamma_n(1) > v\delta^{1/\alpha_n}/t\right) \exp\left(-t\right) \\ &\le C/\delta \int_0^1 dv v^{-\alpha_n} \int_0^\infty dt t^{\alpha_n} \exp\left(-t\right) \\ &\le C/\left(\delta\left(1-\alpha_n\right)\right), \qquad (2.3.27)\end{aligned}$$

which is finite. Similarly we can show that

$$(I) \le a_n/n \sum_{k=1}^{n-1} \left(F_n\left(v_k\right) - \mathbb{E}\left[F_n\left(v_k\right)\right]\right) + 2\,. \quad (2.3.28)$$

Now we bound for $n \in \mathbb{N}$ and $k \in \{1, \dots, n\}$, $\mathbb{P}$-a.s., the term $F_n\left(v_k\right) - \mathbb{E}F_n\left(v_k\right)$, by constants $t_{n,k}$, i.e. we choose $t_{n,k}$'s such that

$$\begin{aligned}&\mathbb{P}\left(\exists k \le n-1 : a_n\left(F_n\left(v_k\right) - \mathbb{E}F_n\left(v_k\right)\right) > t_{n,k}\right) \\ &\le \sum_{k=1}^{n-1} \mathbb{P}\left(a_n\left(F_n\left(v_k\right) - \mathbb{E}F_n\left(v_k\right)\right) > t_{n,k}\right) \qquad \le n n^{-\eta}, \qquad (2.3.29)\end{aligned}$$

for $\eta > 2$. Let $n \ge N^\star$ and $k \in \{1, \dots, n-1\}$. We use (2.3.11) of Proposition 2.10 for $Y_n\left(k\right) = \exp\left(-\frac{k}{n}/\gamma_n(i)\right)$. Choose $a = 2$ and take

$$\bar{b}_{n,k}^2 = \frac{Cn}{4a_n\delta}\Gamma\left(1+\alpha_n\right) 2^{-\alpha_n} \left(\frac{k}{n}\right)^{-\alpha_n}. \quad (2.3.30)$$

Thus, the $t_{n,k}$ should on the one hand satisfy

$$t_{n,k} \le \frac{C}{8\delta}\, \Gamma\left(1+\alpha_n\right) 2^{-\alpha_n} \left(\frac{k}{n}\right)^{-\alpha_n}, \quad (2.3.31)$$

and on the other hand they must be such that

$$\exp\left(-t_{n,k}^2 (2k)^{\alpha_n} \delta n (4Cn^{1+\alpha_n}\Gamma\left(1+\alpha_n\right))^{-1}\right) \le n^{-\eta} \quad (2.3.32)$$

Choosing

$$t_{n,k} = \left(4C\eta\Gamma(1+\alpha_n) a_n n^{\alpha_n} \log n (nk\delta 2^{\alpha_n})^{-1}\right)^{1/2}, \quad (2.3.33)$$

we see that (2.3.31) is satisfied if and only if

$$\frac{256\delta}{C}\eta a_n/n\log n \leq \Gamma\left(1+\alpha_n\right)\left(n/2k\right)^{\alpha_n} . \tag{2.3.34}$$

Since $\Gamma\left(1+\alpha_n\right) > 4/5$ and $n/(2k) > 1/2$ for $k \leq n-1$ this is in particular satisfied if

$$\widetilde{C}a_n/n\log n \leq 2^{-\alpha_n} , \tag{2.3.35}$$

for $\widetilde{C} = 320\eta\delta/C$. By assumption (i) we know that $\widetilde{C}\left(\log n\right)\frac{a_n}{n} \to 0$, whereas $2^{-\alpha_n} \to 1$. Thus, for n large enough, both requirements are satisfied for $t_{n,k}$ as in (2.3.33). It follows that there exist $\eta > 2$ and $t_{n,k}$ as desired in (2.3.29). Hence, by Borel-Cantelli Lema, we get the following $\mathbb{P}$-a.s. bound

$$\begin{aligned}
(I) &\leq 1/n\textstyle\sum_{k=1}^{n-1} t_{n,k}+2 \\
&= 2/n\left(C\Gamma\left(1+\alpha_n\right)\eta a_n\log n(n\delta 2^{\alpha_n})^{-1}\right)^{1/2}\textstyle\sum_{k=1}^{n-1}(n/k)^{\alpha_n/2}+2 \\
&\leq 2\left(C\eta a_n\log n(\delta n)^{-1}\right)^{1/2}(1-\alpha_n/2)^{-1}+2 .
\end{aligned} \tag{2.3.36}$$

By assumption (i) we know that for n large enough,

$$(I) \leq (1-\alpha_n/2)+2 . \tag{2.3.37}$$

Collecting the bounds for (I) and (2), we find that, $\mathbb{P}$-a.s.,

$$\begin{aligned}
&\delta^{-1/\alpha_n}\mathcal{E}_{\pi_n}\textstyle\sum_{k=1}^{a_n}\gamma_n(J_n(k))e_{n,k}\mathbb{1}_{\gamma_n(J_n(k))e_{n,k}\leq\delta^{1/\alpha_n}} \\
\leq\ & 3+C/(\delta(1-\alpha_n))+(1-\alpha_n/2).
\end{aligned} \tag{2.3.38}$$

This bound is, taking the α_n^{th} norm, finite as first $n \to \infty$ and then $\delta \to 0$. This finishes the verification of Condition (3).

2.3.1 Proofs of Theorem 2.4 and Theorem 2.5

We have just established that under the assumptions of Theorem 2.4, $\mathbb{P}$-a.s., the conditions of Theorem 2.2 are satisfied for the measure $\nu(u,\infty) = u^{-1}$, $u >$. Thus, Theorem 2.2 implies that, $\mathbb{P}$-a.s., $S_n^{\alpha_n} \overset{J_1}{\Longrightarrow} M$, where M is an extremal process with distribution function $F\left(x\right) = \exp\left(-1/x\right)$. This finishes the proof of Theorem 2.4.

We deduce now from Theorem 2.4 the convergence of the correlation function $\mathcal{C}_n$ as defined in (2.1.30). Note to this end that we have for any interval (a,b)

$$\mathcal{P}_{\pi_n}(X_n(a) = X_n(s), \quad \forall a < s < b) = \mathcal{P}_{\pi_n}(\{\widetilde{S}_n(k), k = 0,1,\ldots\} \cap (a,b) = \emptyset). \tag{2.3.39}$$

Therefore, we may rewrite $\mathcal{C}_n$ for $s,t > 0$ as

$$\begin{aligned}
\mathcal{C}_n(s,t) &= \mathcal{P}_{\pi_n}\left(\left\{\widetilde{S}_n(k), k = 0,1,\ldots\right\} \cap (c_n s^{1/\alpha_n}, c_n(t+s)^{1/\alpha_n}) = \emptyset\right) \\
&= \mathcal{P}_{\pi_n}\left(\{\left(c_n^{-1}\widetilde{S}_n(k)\right)^{\alpha_n}, k = 0,1,\ldots\} \cap (s,t+s) = \emptyset\right) \\
&= \mathcal{P}_{\pi_n}\left(\{(S_n(v))^{\alpha_n}, v \geq 0\} \cap (s,t+s) = \emptyset\right) .
\end{aligned} \tag{2.3.40}$$

Our aim is to use the weak convergence of $S_n^{\alpha_n}$ to determine the limit of $\mathcal{C}_n(s,t)$. Thus, we write the event inside the probability with the help of a continuous function, namely the overshoot function, which we now define. For $Y \in D[0,\infty)$ let $\mathcal{L}_v(Y)$ be the time of the first passage to the level

$v > 0$ and let $D_v(Y) = Y(\mathcal{L}_v(Y))$ be the first visit to the set $\{Y(u), u > 0\}$ after $\mathcal{L}_v(Y)$. The overshoot function $\chi_v(Y)$ is defined by $\chi_v(Y) \equiv Y(\mathcal{L}_v(Y)) - v$. We have

$$\mathcal{C}_n(s,t) = \mathcal{P}_{\pi_n}\big(\chi_t(S_n^{\alpha_n}) \geq t+s\big). \tag{2.3.41}$$

Theorem 13.6.5 in [63] states that the overshoot function is a continuous functional on $D[0,\infty)$ equipped with Skorohod's J_1 topology with respect to Lévy motions with $\mathcal{P}$-a.s. diverging paths. Since M has $\mathcal{P}$-a.s. diverging paths, we find that, $\mathbb{P}$-a.s., for all $s, t > 0$

$$\lim_{n\to\infty} \mathcal{C}_n(s,t) = \mathcal{P}\big(\chi_t(M) \geq t+s\big) = \mathcal{P}\big(\mathcal{R}_M \cap (t,t+s) = \emptyset\big), \tag{2.3.42}$$

where $\mathcal{R}_M$ denotes the range of M. By Proposition 4.8 in [56] we know that the range of M is the range of a Poisson random measure ξ' with intensity measure $\nu'(u,\infty) = \log u$. Thus, we get that

$$\mathcal{P}(\mathcal{R}_M \cap (t,t+s) = \emptyset) = \mathcal{P}(\xi'(t,t+s) = 0) = e^{-\nu'(t,t+s)} = \tfrac{t}{t+s}, \tag{2.3.43}$$

which is as desired and the proof of Theorem 2.5 is finished.

2.4 Application to the REM

We show that the conditions of Theorem 2.2 are satisfied $\mathbb{P}$-a.s. for the sequences a_n, c_n, and α_n as in Theorem 2.6 and the block length $\theta_n \equiv 1$.

2.4.1 Verification of Condition (1) and (2)

In this section we establish that, $\mathbb{P}$-a.s. Condition (1) and (2) are satisfied. This proof comes in two steps, which we now describe. First we show that $\mathcal{E}_{\pi_n}\nu_n^{J,t}$ converges $\mathbb{P}$-a.s. to t/u, respectively that $\mathcal{E}_{\pi_n}\sigma_n^{J,t}$ converges $\mathbb{P}$-a.s. to zero. This is done in Lemma 2.11. In the second step we show that, $\mathbb{P}$-a.s., $\nu_n^{J,t}$ and $\sigma_n^{J,t}$ concentrate around their means in $\mathcal{P}_{\mu_n}$ probability. This concentration result is contained in Lemma 2.12. Let us present the above mentioned lemmata, but postpone their proofs. To simplify notation, we define for $n \in \mathbb{N}$ and $u > 0$

$$\nu_n(u,\infty) = a_n 2^{-n} \sum_{x\in\mathcal{V}_n} \exp\left(-u^{1/\alpha_n}/\gamma_n(x)\right) \tag{2.4.1}$$

$$\sigma_n(u,\infty) = a_n 2^{-n} \sum_{y\in\mathcal{V}_n} \left(1/n \sum_{x\sim y} \exp\left(-u^{1/\alpha_n}/\gamma_n(x)\right)\right)^2, \tag{2.4.2}$$

which are random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. The quantity $\nu_n(u,\infty)$ is the expected value w.r.t. $\mathcal{E}_{\pi_n}$ of $\nu_n^{J,1}$ and the same is true for σ_n. To see this, we use the fact that π_n is stationary and reversible and find for $y \in \mathcal{V}_n$ that

$$E_{\pi_n}\left((k_n(t))^{-1}\textstyle\sum_{k=1}^{\lfloor a_n t\rfloor} \mathbb{1}_{J_n(k-1)=y}\right) = (k_n(t))^{-1}\textstyle\sum_{k=1}^{\lfloor a_n t\rfloor} \pi_n(y) = \pi_n(y). \tag{2.4.3}$$

Therefore, we see by (2.4.1) that

$$\mathcal{E}\left(\nu_n^{J,t}(u,\infty)\right) = (k_n(t)/a_n)\nu_n(u,\infty), \tag{2.4.4}$$

and by (2.4.2) that

$$E\left(\sigma_n^{J,t}(u,\infty)\right) = (k_n(t)/a_n)\sigma_n(u,\infty). \tag{2.4.5}$$

The $\mathbb{P}$-a.s. convergence of ν_n and σ_n follows from the following lemma.

Lemma 2.11. *There exists Ω_1^τ with $\mathbb{P}(\Omega_1^\tau) = 1$ such that on Ω_1^τ we have for all $u > 0$*

$$\lim_{n\to\infty} |\nu_n(u,\infty) - 1/u| = 0\,, \tag{2.4.6}$$

$$\lim_{n\to\infty} \sigma_n(u,\infty) = 0\,. \tag{2.4.7}$$

Having (2.4.4) and (2.4.5) in mind, the following lemma shows that $\nu_n^{J,t}$, respectively $\sigma_n^{J,t}$ concentrate in $\mathcal{P}_{\pi_n}$-probability around their mean. This lemma is the pendant to Proposition 4.1 in [36] in the case $\alpha_n \to 0$.

Lemma 2.12. *Define $\rho_n \equiv \frac{3(\log 2)\alpha_n}{2\sqrt{2\pi}}$ and $\theta_n \equiv 2\left\lceil \frac{3(n-1)\log 2}{2|\log(1-2/n)|}\right\rceil$. There exists a sequence $\Omega_{n,0}$ with $\mathbb{P}((\Omega_{n,0})^c) \le n^{-p}$ for $p > 1$ and n large enough and such that on $\Omega_{n,0}$ for all $u,t,\varepsilon > 0$ we have*

$$\mathcal{P}_{\pi_n}\left(|\nu^{J,t}(u,\infty) - \tfrac{\lfloor a_n t\rfloor}{a_n}\nu_n(u,\infty)| > \varepsilon\right) \le \Theta_n(t,u)/\varepsilon^2. \tag{2.4.8}$$

where $\Theta_n(t,u)$ is given by

$$\Theta_n(t,u) = t\left[t(\nu_n(u,\infty))^2/2^n + \sigma_n(u,\infty) + C\nu_n(u,\infty)/n^2 + \rho_n(\mathbb{E}[\nu_n(u,\infty)])^2\right], \tag{2.4.9}$$

for $C \in (0,\infty)$. Moreover, for $\varepsilon' > 0$ we have

$$\mathcal{P}_{\pi_n}\left(\sigma_n^{J,t}(u,\infty) > \varepsilon'\right) \le \lfloor a_n t\rfloor/(a_n\varepsilon')\sigma_n(u,\infty)\,. \tag{2.4.10}$$

Verification of Condition (1) and Condition (2). Lemma 2.12 implies that, $\mathbb{P}$-a.s., we have for n large enough for all $u,t,\varepsilon > 0$ that

$$\begin{aligned} &\mathcal{P}_{\pi_n}\left(\left|\nu_n^{J,t}(u,\infty) - t/u\right| > \varepsilon\right) \\ \le\ &\mathcal{P}\left(\left|\nu_n^{J,t}(u,\infty) - k_n(t)/a_n\nu_n(u,\infty)\right| > \varepsilon\right) + \mathbb{1}_{|k_n(t)/a_n\nu_n(u,\infty)-t/u|>\varepsilon} \\ \le\ &\Theta_n(t,u)/\varepsilon^2 + \mathbb{1}_{|k_n(t)/a_n\nu_n(u,\infty)-t/u|>\varepsilon}. \end{aligned} \tag{2.4.11}$$

By Lemma 2.11 we know that, $\mathbb{P}$-a.s., for all $u > 0$ $\nu_n(u,\infty) \to u^{-1}$, and $\sigma_n(u,\infty) \to 0$ as $n \to \infty$. Therefore, we know that $\mathbb{P}$-a.s., for all $u,t > 0$, $\Theta_n(t,u)$ vanishes and the second summand in (2.4.11) equals zero for n large enough. In the same way we show that the following tends, $\mathbb{P}$-a.s., to zero

$$\begin{aligned} &\mathcal{P}\left(\sigma_n^{J,t}(u,\infty) > \varepsilon\right) \\ \le\ &\mathcal{P}\left(\left|\sigma_n^{J,t}(u,\infty) - k_n(t)/a_n\sigma_n(u,\infty)\right| > \varepsilon\right) + \mathbb{1}_{k_n(t)/a_n\sigma_n(u,\infty)>\varepsilon}. \end{aligned} \tag{2.4.12}$$

This shows that $\mathbb{P}$-a.s. Condition (1) and Condition (2) are satisfied. □

Let us now present the proof of Lemma 2.11. It uses the following estimates on the distribution of the $\gamma_n(x)$, $x \in \mathcal{V}_n$.

Lemma 2.13. *For $1 \in \mathcal{V}_n$ and every $u,v > 0$*

(i) $\lim_{n\to\infty} a_n\mathbb{P}\left(\gamma_n(1) > u^{1/\alpha_n}\right) = u^{-1}$,

(ii) $\lim_{n\to\infty} \left|a_n\mathbb{P}\left(\gamma_n(1) > u^{1/\alpha_n}v\right) - u^{-1}v^{-\alpha_n}\right| = 0$,

(iii)

$$a_n\mathbb{P}\left(\gamma_n\left(1\right)>u\right)\leq\begin{cases}u^{-\alpha_n}\,, & u\geq 1\\ 2u^{-\alpha_n}\,, & u\in[c_n^{-1/2},1)\\ \sqrt{2\pi}u^{-2\alpha_n}\,, & u\in(c_n^{-1},c_n^{-1/2})\\ \sqrt{2\pi}u^{-\alpha_n}\,, & u\in(0,c_n^{-1}].\end{cases}\tag{2.4.13}$$

In particular, we have for $u>0$

$$\lim_{n\to\infty}a_n\mathbb{E}\left[\exp\left(-u^{1/\alpha_n}/\gamma_n\left(1\right)\right)\right]=u^{-1}.\tag{2.4.14}$$

Proof of Lemma 2.13. We use the well-known first order approximations of the normal distribution

$$\mathbb{P}\left(\mathcal{N}\left(0,1\right)>u\right)=(\sqrt{2}\pi u)^{-1}\exp\left(-u^2/2\right)\left(1+o\left(1\right)\right),\quad u\gg 1\tag{2.4.15}$$

$$\mathbb{P}\left(\mathcal{N}\left(0,1\right)>u\right)\leq(\sqrt{2}\pi u)^{-1}\exp\left(-u^2/2\right),\quad u>0\,.\tag{2.4.16}$$

First we show the claim (i), i.e. we show that for $u>0$, $1\in\mathcal{V}_n$

$$\mathbb{P}\left(\gamma_n\left(1\right)>u^{1/\alpha_n}\right)=\mathbb{P}\left(H_n\left(1\right)>(\sqrt{n}\beta)^{-1}\left(\log c_n+\alpha_n^{-1}\log u\right)\right)\overset{n\to\infty}{\longrightarrow}u^{-1},\tag{2.4.17}$$

where we recall that $H_n\left(1\right)\sim\mathcal{N}\left(0,1\right)$. In order to apply (2.4.15), it is necessary to show that for $u>0$, the following is an increasing sequence

$$u_n=(\sqrt{n}\beta)^{-1}\left(\log c_n+\alpha_n^{-1}\log u\right).\tag{2.4.18}$$

By (A) - (C) we have

$$a_n=\sqrt{2\pi n}\beta\alpha_n\exp\left(\tfrac{1}{2}\alpha_n^2n\beta^2\right)\geq\sqrt{2\pi\log n}\exp\left(\tfrac{7+\eta}{2}\log n\right)\sim\sqrt{\log n}\,n^{\frac{7+\eta}{2}}\tag{2.4.19}$$

$$c_n=\exp\left(\alpha_nn\beta^2\right)\geq\exp\left(\sqrt{(7+\eta)\left(\log n\right)\ n}\beta\right)\,,\tag{2.4.20}$$

which both diverge as $n\to\infty$. Thus, for $u\geq 1$, since $u_n>0$, u_n diverges because

$$(\sqrt{n}\beta)^{-1}\log c_n=\alpha_n\sqrt{n}\beta\geq((7+\eta)\log n)^{1/2}\to\infty.\tag{2.4.21}$$

Moreover, for $u\in(0,1)$, we know that $u_n\leq 0$ for some $n\in\mathbb{N}$. But for n large enough such that $u>n^{-(7+\eta)}$ we get by (A)

$$\begin{aligned}u_n&=(\sqrt{n}\beta)^{-1}\left(\alpha_nn\beta^2+\alpha_n^{-1}\log u\right)\\&\geq((7+\eta)\log n)^{1/2}+\log u((7+\eta)\log n)^{-1/2}\\&>((7+\eta)\log n)^{1/2}-((7+\eta)\log n)((7+\eta)\log n)^{-1/2}=0\,.\end{aligned}\tag{2.4.22}$$

Since u_n is increasing in n, it follows in particular that $u_n\to\infty$. Thus, we find for $u>0$ and n large enough by (2.4.15) and (A) - (C)

$$\begin{aligned}&a_n\mathbb{P}\left(\gamma_n\left(1\right)>u^{1/\alpha_n}\right)\\&=\frac{a_n(n/(2\pi n))^{1/2}\beta}{\alpha_n^{-1}\log u+\log c_n}\exp\left(-\tfrac{\log c_n\log u}{n\beta^2\alpha_n}-\tfrac{1}{2n\beta^2}\left(\tfrac{(\log u)^2}{\alpha_n^2}+(\log c_n)^2\right)\right)\\&=((1+\log u/(\alpha_n\log c_n))u)^{-1}\exp\left(-\tfrac{(\log u)^2}{2\alpha_n\log c_n}\right)\,.\end{aligned}\tag{2.4.23}$$

Note that $\alpha_n \log c_n = \alpha_n^2 n\beta^2 \to \infty$ (cf. (2.4.19)), and hence $\log u(\alpha_n \log c_n) \to 0$, respectively $(\log u)^2/(2\alpha_n \log c_n) \to 0$, which proves (i). By using similar arguments, one show the claim of (ii).

Let us establish (iii). We consider each case separately.

Case 1*:* Let $u \geq 1$, i.e. in particular $u_n > 0$. Hence we my apply (2.4.16) to find for $n \in \mathbb{N}$ that

$$\begin{aligned} a_n\,\mathbb{P}\left(\gamma_n\left(1\right) > u\right) &\leq \frac{a_n\sqrt{n}\beta}{\sqrt{2\pi}\left(\log c_n + \log u\right)} \exp\left(-\tfrac{1}{2n\beta^2}\left(\log c_n + \log u\right)^2\right) \\ &= u^{-\alpha_n}(1 + \log u/\log c_n)^{-1} \exp\left(-(\log u)^2/(2n\beta^2)\right) \\ &\leq u^{-\alpha_n}, \end{aligned} \tag{2.4.24}$$

as desired.

Case 2*:* Let u and n be such that $u \in [c_n^{-1/2}, 1)$. Then $\log u < 0$, but $(\log c_n + \log u) > 0$. Hence, by similar calculations as in the first case we get that

$$a_n\mathbb{P}\left(\gamma_n\left(1\right) > u\right) \leq u^{-\alpha_n}(1 + \tfrac{\log u}{\log c_n})^{-1} \leq u^{-\alpha_n}(1 - \tfrac{1/2\log c_n}{\log c_n})^{-1} \leq 2u^{-\alpha_n}, \tag{2.4.25}$$

as claimed in (iii).

Case 3*:* Let u and n be such that $u \in (c_n^{-1}, c_n^{-1/2})$. By (B) and (C) we know that

$$\begin{aligned} a_n\,\mathbb{P}\left(\gamma_n\left(1\right) > u\right) \leq a_n &= \sqrt{2\pi}\exp\left(\tfrac{1}{2}\alpha_n^2 n\beta^2\right)\sqrt{n\beta^2\alpha_n^2} \\ &\leq \sqrt{2\pi}\exp\left(\alpha_n^2 n\beta^2\right) \\ &= \sqrt{2\pi}c_n^{\alpha_n}. \end{aligned} \tag{2.4.26}$$

Since $u \in (c_n^{-1}, c_n^{-1/2})$ this implies that

$$a_n\,\mathbb{P}\left(\gamma_n\left(1\right) > c_n u\right) \leq \sqrt{2\pi}u^{-2\alpha_n}. \tag{2.4.27}$$

Case 4*:* If u, n are s.t. $u \in \left(0, c_n^{-1}\right)$, then (iii) follows by the same arguments as in the third case and the fact that $c_n^{\alpha_n} \leq u^{-\alpha_n}$.

To conclude the proof, we show that (2.4.14) holds. Let $u > 0$. Then, as in (2.3.7), we know for $1 \in \mathcal{V}_n$ that

$$a_n\,\mathbb{E}\left[\exp\left(-u^{1/\alpha_n}/\gamma_n\left(1\right)\right)\right] = a_n \textstyle\int_0^\infty dy e^{-y}\mathbb{P}\left(\gamma_n\left(1\right) > u^{1/\alpha_n}/y\right). \tag{2.4.28}$$

By (ii) of Lemma 2.13 we know for all $y > 0$ that

$$\lim_{n\to\infty}\left|e^{-y}\mathbb{P}\left(\gamma_n\left(1\right) > u^{1/\alpha_n}/y\right) - e^{-y}y^{\alpha_n}u^{-1}\right| = 0. \tag{2.4.29}$$

Moreover, by (iii) of Lemma 2.13 we have that

$$\begin{aligned} e^{-y}\mathbb{P}\left(\gamma_n\left(1\right) > u^{1/\alpha_n}/y\right) &\leq e^{-y}\sqrt{2\pi}\left(u^{-1}y^{\alpha_n} + u^{-2}y^{2\alpha_n}\right) \\ &\leq e^{-y}\sqrt{2\pi}\max\left\{u^{-1}, u^{-2}\right\}\left(\mathbb{1}_{\{y<1\}} + y\mathbb{1}_{\{1\leq y\}}\right), \end{aligned} \tag{2.4.30}$$

for n such that $\alpha_n < 1/2$. Therefore, we may apply dominated convergence to obtain that

$$a_n\,\mathbb{E}\left[\exp\left(-u^{1/\alpha_n}/\gamma_n\left(1\right)\right)\right] \sim u^{-1}\textstyle\int_0^\infty dy e^{-y}\,y^{\alpha_n} = u^{-1}\Gamma\left(1 + \alpha_n\right), \tag{2.4.31}$$

which tends as $n \to \infty$ to u^{-1}. The proof of Lemma 2.13 is finished. $\square$

Now we prove Lemma 2.11. To this end the following remark is helpful.

Remark. Our choice of a_n satisfies for n large enough,

$$2^{-n}a_n \leq \sqrt{2\pi}\exp\left(n\ \left(\alpha_n^2\beta^2 - \log 2\right)\right) \leq \exp\left(-n/2\right), \tag{2.4.32}$$

where we used the fact that $\alpha_n \to 0$. In particular we know that $\sum_{n\in\mathbb{N}} 2^{-n}a_n < \infty$.

Proof of Lemma 2.11. We split the proof into two parts. First we prove the statement for ν_n, i.e. (2.4.6) and in the second part the claim for σ_n, i.e. (2.4.7).
Claim 1*:* We show that there exists $\Omega_{1,\nu}$ with $\mathbb{P}\left(\Omega_{1,\nu}\right) = 1$ and such that on $\Omega_{1,\nu}$ we have

$$\lim_{n\to\infty} a_n \left|2^{-n}\sum_{x\in\mathcal{V}_n}\exp\left(-u^{1/\alpha_n}/\gamma_n\left(x\right)\right) - \mathbb{E}\left[\exp\left(-u^{1/\alpha_n}/\gamma_n\left(1\right)\right)\right]\right| = 0. \tag{2.4.33}$$

Since we know by Lemma 2.13 that

$$\lim_{n\to\infty} a_n\mathbb{E}\left[\exp\left(-u^{1/\alpha_n}/\gamma_n\left(1\right)\right)\right] = u^{-1}, \tag{2.4.34}$$

we get by (2.4.33), that for $\varepsilon > 0$ we have on $\Omega_{1,\nu}$ for n large enough

$$\begin{aligned} &|\nu_n(u,\infty) - u^{-1}| \\ \leq\ & a_n\left|2^{-n}\sum_{x\in\mathcal{V}_n}\exp\left(-u^{1/\alpha_n}/\gamma_n\left(x\right)\right) - \mathbb{E}\left[\exp\left(-u^{1/\alpha_n}/\gamma_n\left(1\right)\right)\right]\right| \\ +\ & \left|a_n\mathbb{E}\left[\exp\left(-u^{1/\alpha_n}/\gamma_n\left(1\right)\right)\right] - u^{-1}\right| \qquad\qquad < 2\varepsilon, \end{aligned} \tag{2.4.35}$$

showing that $\mathbb{P}$-a.s. ν_n converges to u^{-1}. Let us now establish the claim of (2.4.33). We apply (2.3.11) of Proposition 2.10. For $n \in \mathbb{N}$ and $x \in \mathcal{V}_n$ we define the random variable $Y_n\left(x\right) = \exp\left(-u^{1/\alpha_n}/\gamma_n\left(x\right)\right) - \mathbb{E}\left[\exp\left(-u^{1/\alpha_n}/\gamma_n\left(1\right)\right)\right]$ and take $a = 2$. We choose $\bar{b}^2$ by the following calculation

$$\sum_{x\in\mathcal{V}_n}\mathbb{E}\left[\left(Y_n\left(x\right)\right)^2\right] \leq 2^n\mathbb{E}\left[\exp\left(-u^{1/\alpha_n}/\gamma_n\left(1\right)\right)\right] \leq 2^n/a_n\mathbb{E}\left[\nu_n\left(u,\infty\right)\right] = \bar{b}^2\ . \tag{2.4.36}$$

Let $\varepsilon > 0$ and set $t = \varepsilon 2^n/a_n$. We check whether $t \leq \bar{b}^2/(2a)$, which is equivalent to

$$\varepsilon \leq \mathbb{E}\left[\nu_n\left(u,\infty\right)\right]/4\ . \tag{2.4.37}$$

From the proof of (2.4.14) we know that $|\mathbb{E}\left[\nu_n\left(u,\infty\right)\right] - \Gamma\left(1+\alpha_n\right)/u| < \varepsilon$ for n large enough. Therefore, (2.4.37) is in particular satisfied for ε such that

$$\varepsilon \leq \Gamma\left(1+\alpha_n\right)/(5u)\ . \tag{2.4.38}$$

For such ε we may apply (2.3.11) of Proposition 2.10 to find for n large enough that

$$\begin{aligned} &\mathbb{P}\left(a_n\left|2^{-n}\sum_{x\in\mathcal{V}_n}\exp\left(-u^{1/\alpha_n}/\gamma_n\left(x\right)\right) - \mathbb{E}\left[\exp\left(-u^{1/\alpha_n}/\gamma_n\left(1\right)\right)\right]\right| > \varepsilon\right) \\ \leq\ & \exp\left(-\varepsilon^2/(4\mathbb{E}\left[\nu_n\left(u,\infty\right)\right])2^n/a_n\right) \\ \leq\ & \exp\left(-\varepsilon^2/(4(\varepsilon+1/u))2^n/a_n\right). \end{aligned} \tag{2.4.39}$$

By 2.4.32 we know in particular that this is summable in n and the claim of (2.4.6) follows from Borel-Cantelli Lemma.

Claim 2*:* We first show that $\mathbb{E}\left[\sigma_n\left(u,\infty\right)\right] \to 0$ for all $u > 0$. Then we use a second order Chebyshev inequality to prove that $\sigma_n(u,\infty)$ concentrates for $u > 0$, $\mathbb{P}$-a.s. around its mean. Let $u > 0$. We have for $1 \in \mathcal{V}_n$ that

$$\begin{aligned}
\mathbb{E}\left[\sigma_n\left(u,\infty\right)\right] &= a_n 2^{-n}\mathbb{E}\Big[\textstyle\sum_{x\in\mathcal{V}_n}\Big(\sum_{y\sim x} 1/n\exp(-u^{1/\alpha_n}/\gamma_n\left(y\right))\Big)^2\Big] \\
&= a_n\mathbb{E}\Big[\Big(\textstyle\sum_{y\sim 1} 1/n\exp(-u^{1/\alpha_n}/\gamma_n\left(y\right))\Big)^2\Big] \\
&= a_n/n\mathbb{E}\Big[\exp(-2u^{1/\alpha_n}/\gamma_n\left(y\right))\Big] + a_n\tfrac{n-1}{n}\mathbb{E}^2\Big[\exp(-u^{1/\alpha_n}/\gamma_n\left(y\right))\Big] \\
&\leq \mathbb{E}\left[\nu_n\left(u,\infty\right)\right]/n + \mathbb{E}^2\left[\nu_n\left(u,\infty\right)\right]/a_n \\
&\sim \Gamma\left(1+\alpha_n\right)/(nu) + \Gamma^2\left(1+\alpha_n\right)/(a_n u^2),
\end{aligned} \tag{2.4.40}$$

where we used in the third step the fact that

$$a_n\mathbb{E}\left[\left(\gamma_n^u\left(1\right)\right)^2\right] = a_n 2^{-n}\textstyle\sum_{x\in\mathcal{V}_n}\mathbb{E}\left[\exp\left(-2u^{1/\alpha_n}/\gamma_n\left(x\right)\right)\right] \leq \mathbb{E}\left[\nu_n\left(u,\infty\right)\right]. \tag{2.4.41}$$

Since the right hand side of (2.4.40) tends to zero as $n\to\infty$, this finishes the proof that $\mathbb{E}\sigma_n\left(u,\infty\right)$ vanishes in the limit. However, the convergence speed is not summable in n and we therefore cannot yet conclude that $\sigma_n(u,\infty)$ tends $\mathbb{P}$-a.s. to zero. Therefore, we use a second order Chebyshev inequality to find for $\varepsilon > 0$ that

$$\mathbb{P}\left(\sigma_n\left(u\right) > \varepsilon\right) \leq \mathbb{E}\left(\sigma_n\right)^2/\varepsilon^2. \tag{2.4.42}$$

Let us calculate $\mathbb{E}\left(\sigma_n\right)^2$. Writing $\gamma_n^u(x) \equiv \exp\left(-u^{1/\alpha_n}/\gamma_n\left(x\right)\right)$, it is given by

$$\begin{aligned}
&\mathbb{E}\left[\left(\sigma_n\left(u,\infty\right)\right)^2\right] \\
&= \left(a_n/2^n\right)^2\textstyle\sum_{x,x'}\mathbb{E}\Big(1/n\sum_{y\sim x}\gamma_n^u\left(y\right)\Big)^2\Big(1/n\sum_{y'\sim x'}\gamma_n^u\left(y'\right)\Big)^2 \\
&= a_n^2/(2^{2n}n^4)\Big\{2^n\mathbb{E}(\textstyle\sum_{x\sim 1}\gamma_n^u\left(x\right))^4 + \sum_{x\neq x'}\mathbb{E}\Big(\sum_{y\sim x}\gamma_n^u\left(y\right)\Big)^2\Big(\sum_{y'\sim x'}\gamma_n^u\left(y'\right)\Big)^2\Big\} \\
&= (I) + (II)\ .
\end{aligned} \tag{2.4.43}$$

First we bound (I). By independence and identical distribution of the γ_n^u's, there are five different types of summands. Either all four random variables are the same, all four different, each two are same, two are same two are different, or three are same and the fourth is different. Thus, we find that

$$\begin{aligned}
(I) &= a_n^2/(2^n n^4)\Big\{n\,\mathbb{E}\left[\left(\gamma_n^u\left(1\right)\right)^4\right] + \tbinom{n}{4}\mathbb{E}^4\left[\gamma_n^u\left(1\right)\right] + \tbinom{n}{3}\mathbb{E}^2\left[\gamma_n^u\left(1\right)\right]\mathbb{E}^2\left[\gamma_n^u\left(1\right)\right] + \\
&+ \tbinom{n}{2}\Big(\mathbb{E}^2\left[\left(\gamma_n^u\left(1\right)\right)^2\right] + \mathbb{E}\left[\left(\gamma_n^u\left(1\right)\right)^3\right]\mathbb{E}\left[\gamma_n^u\left(1\right)\right]\Big)\Big\},
\end{aligned} \tag{2.4.44}$$

which is for n large enough smaller than

$$a_n/2^n\left[\tfrac{\mathbb{E}\left[\nu_n(u,\infty)\right]}{n^3} + \tfrac{\mathbb{E}^4\left[\nu_n(u,\infty)\right]}{a_n^3} + \tfrac{\mathbb{E}\left[\nu_n(u,\infty)\right]\mathbb{E}^2\left[\nu_n(u,\infty)\right]}{a_n^2 n} + \tfrac{2\mathbb{E}^2\left[\nu_n(u,\infty)\right]}{a_n n^2}\right] \leq 4a_n/2^n. \tag{2.4.45}$$

By 2.4.32, (I) is summable in n. To bound (II), we first of all note that there are altogether $2^n\left(2^n-1\right)$ summands in (II), which we bound separately. To this end, we distinguish the following two cases:

Case 1*:* The two sides y, y' have no common neighbors. Notice that this also includes the case when $y \sim y'$. Then each summand contains the following three different types of combinations:

all four random variables are different, each two random variables are same, or two are same and the other two are different. Up to the factor $\frac{a_n^2}{2^{2n}n^4}$, each summand is of the form

$$(n\,(n-1))^2\mathbb{E}(\gamma_n^u\,(1))^4 + n^2\left(\mathbb{E}\,(\gamma_n^u\,(1))^2\right)^2 + n^2\,(n-1)\,(\mathbb{E}\gamma_n^u\,(1))^2\,\mathbb{E}\,(\gamma_n^u\,(1))^2\,. \tag{2.4.46}$$

Case 2*:* When y, y' have common neighbors, then there are exactly two common neighbors, say $x_1, x_2 \in \mathcal{V}_n$. In this case, the summand looks different, because x_1 and/or x_2 might appear in the sum dependent on y as well as in the one dependent on y'. The combinations which additionally appear are the following: all four are either x_1 or x_2, two are x_1 and two are x_2, two are either x_1 or x_2 and three are either x_1 or x_2 and the fourth is different. Therefore, we get in addition

$$\begin{aligned} &2\mathbb{E}\left[(\gamma_n^u\,(1))^4\right] + \tbinom{4}{2}\mathbb{E}^2\left[(\gamma_n^u\,(1))^2\right] + 2\tbinom{4}{2}\,(n-1)\,(n-2)\,\mathbb{E}\left[(\gamma_n^u\,(1))^2\right]\mathbb{E}^2\left[\gamma_n^u\,(1)\right] \\ + \quad &2\tbinom{4}{3}\,(n-1)\,\mathbb{E}\left[(\gamma_n^u\,(1))^3\right]\mathbb{E}\left[\gamma_n^u\,(1)\right], \end{aligned} \tag{2.4.47}$$

which appear in (II) exactly $2^n n\,(n-1)$ times. Of course, also the combinations as described in the first case appear in each such summand, but they appear less often than described in (2.4.46). To construct a bound, it suffices to say that there are less than 2^{2n} terms of the form (2.4.46) and exactly $2^n n\,(n-1)$ terms of the form (2.4.47). Hence,

$$\begin{aligned} (II) \quad \leq \quad & \tfrac{\mathbb{E}^4[\nu_n(u,\infty)]}{a_n^2} + \tfrac{\mathbb{E}^2[\nu_n(u,\infty)]}{n^2} + \tfrac{\mathbb{E}^3[\nu_n(u,\infty)]}{a_n n} + \tfrac{a_n}{2^n}\tfrac{2\mathbb{E}[\nu_n(u,\infty)]}{n^2} \\ + \quad & \tfrac{6\,\mathbb{E}^2[\nu_n(u,\infty)]}{2^n n^2} + \tfrac{12\,\mathbb{E}^3[\nu_n(u,\infty)]}{2^n a_n} + \tfrac{8\,\mathbb{E}^2[\nu_n(u,\infty)]}{2^n n}. \end{aligned} \tag{2.4.48}$$

We bound $\mathbb{E}\left[\nu_n\,(u,\infty)\right] \leq C/u$ for n large enough and $C \in (0,\infty)$. Since we know by (2.4.19) that $a_n \geq n^{(7+\eta)/2}$, (II) is smaller than c'/n^2 for $c' \in (0,\infty)$. Together with (2.4.45) this finishes the proof of (2.4.7). □

We prove now the concentration results for $\nu_n^{J,t}$ and $\sigma_n^{J,t}$.

Proof of Lemma 2.12. Lemma 2.12 is a version of Proposition 4.1 in [36]. More precisely, Proposition 4.1 in [36] states that for $\alpha_n = \alpha \in (0,1)$ and a sequence $\rho_n \to 0$ there exists a sequence $(\Omega_{n,0})_{n\in\mathbb{N}}$ such that $\mathbb{P}\left((\Omega_{n,0})^c\right) \leq \frac{\theta_n}{\rho_n a_n}$ and on $\Omega_{n,0}$ both, (2.4.8) and (2.4.10) hold for all $t > 0$, $u > 0$. Since the proof also works for sequences $\alpha_n \to 0$ we do not present it here. It remains to show that our choice of ρ_n implies that there exists $p > 1$ such that $\mathbb{P}\left((\Omega_{n,0})^c\right) \leq n^{-p}$. Notice that, since $\alpha_n \to 0$, we have $\rho_n \to 0$. Moreover, by assumption (A), there exists $\varepsilon > 0$ such that $\alpha_n \geq \left((7+\eta)\,(\log n)/(n\beta^2)\right)^{1/2}$. Let $p = 1 + \frac{\eta}{2}$ and observe that

$$\theta_n = 2\left\lceil 3\log 2\,(n-1)\,/(2|\log\,(1-2/n)\,|)\right\rceil \leq 3\log 2/2n^2, \tag{2.4.49}$$

and hence

$$\theta_n/(\rho_n a_n) \leq n^2\sqrt{2\pi}/(a_n\alpha_n). \tag{2.4.50}$$

By (B), the second term is less or equal to n^{-p} if and only if

$$\log\alpha_n^2 + 1/2\log n + 1/2\alpha_n^2 n\beta^2 + \log\beta \geq (2+p)\log n, \tag{2.4.51}$$

which by (A) in particular is satisfied if

$$\begin{aligned} &\log\left((7+\eta)\log n\right) - \log\beta + (3+\eta/2)\log n \geq (2+p)\log n \\ \Leftrightarrow \quad &\log\left((7+\eta)\log n\right) - \log\beta \geq 0\,, \end{aligned} \tag{2.4.52}$$

which is true for n large enough. This finishes the proof of Lemma 2.12. □

2.4.2 Verification of Condition (0)

Since $\mu_n = \pi_n$, Condition (0) states in this setting that we have, $\mathbb{P}$-a.s.

$$\lim_{n\to\infty} \sum_{x\in\mathcal{V}_n} 2^{-n} \exp(-v^{1/\alpha_n}/\gamma_n(x)) = 0. \tag{2.4.53}$$

With the notation introduced in (2.4.1), this can be reformulated as follows, $\mathbb{P}$-a.s., we have that

$$\lim_{n\to\infty} a_n^{-1} \nu_n(v,\infty) = 0. \tag{2.4.54}$$

But by Section 2.4.1 we know that for all $\varepsilon > 0$ we have $\mathbb{P}$-a.s. for n large enough that

$$\nu_n(v,\infty) \le (1+\varepsilon)/u, \tag{2.4.55}$$

impliying that (2.4.54) is, $\mathbb{P}$-a.s., satisfied. This finishes the verification of Condition (0).

2.4.3 Verification of Condition (3)

We proceed as in the verification of this condition for Bouchaud's trap model on the complete graph. Namely, as in (2.3.22) and (2.3.23) we show that, $\mathbb{P}$-a.s.

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \left(\sum_{k=1}^{a_n} \int_0^1 dv \mathcal{P}\left(\tau_n\left(J_n\left(k\right)\right) e_{n,k} > v c_n \delta^{1/\alpha_n}\right)\right)^{\alpha_n} < \infty \tag{2.4.56}$$

More precisely, we show that for n large enough and fixed $\delta > 0$, the term underneath the brackets in (2.4.56) is at most polynomial in α_n and depends on δ. Since the $e_{n,k}$'s are independent of J_n, since J_n has a stationary initial distribution, and since the $e_{n,k}$'s are i.i.d. we know that

$$\begin{aligned}
&\sum_{k=1}^{a_n} \int_0^1 du \mathcal{P}\left(\gamma_n\left(J_n\left(k\right)\right) e_{n,k} > \delta^{1/\alpha_n} u\right) \\
&= \sum_{k=1}^{a_n} \int_0^1 du \sum_{x\in\mathcal{V}_n} \mathcal{P}\left(\gamma_n\left(x\right) e_{n,k} > \delta^{1/\alpha_n} u, J_n\left(k\right) = x\right) \\
&= \sum_{k=1}^{a_n} \int_0^1 du \sum_{x\in\mathcal{V}_n} \pi_n\left(x\right) \mathcal{P}\left(\gamma_n\left(x\right) e_{n,k} > \delta^{1/\alpha_n} u\right) \\
&= a_n \int_0^1 du \sum_{x\in\mathcal{V}_n} 2^{-n} \exp\left(-\delta^{1/\alpha_n} u/\gamma_n\left(x\right)\right) \qquad = a_n \int_0^1 du F_n\left(u\right),
\end{aligned} \tag{2.4.57}$$

where we write for $u \in [0,1]$, $F_n\left(u\right) = 2^{-n} \sum_{x\in\mathcal{V}_n} \exp\left(-\delta^{1/\alpha_n} u/\gamma_n\left(x\right)\right)$. As in (2.3.26), we bound the integral in (2.4.57) by

$$\begin{aligned}
& a_n/2^n + a_n \sum_{k=1}^{2^n-1} 2^{-n} \left(F_n\left(\tfrac{k}{2^n}\right) - \mathbb{E}\left[F_n\left(\tfrac{k+1}{2^n}\right)\right]\right) + a_n \int_0^1 du \mathbb{E}\left[F_n\left(u\right)\right] \\
= \; & a_n/2^n + \left(I\right) + \left(II\right).
\end{aligned} \tag{2.4.58}$$

Let us first bound (II).We again use Lemma 2.13 to obtain

$$\begin{aligned}
(II) &= \int_0^1 du \int_0^\infty dt\, a_n \mathbb{P}\left(\gamma_n\left(1\right) > \delta^{1/\alpha_n} u/t\right) \exp\left(-t\right) \\
&\le \sqrt{2\pi} \int_0^1 du \left(\delta^{-2} u^{-2\alpha_n} \Gamma\left(1+2\alpha_n\right) + \delta^{-1} u^{-\alpha_n} \Gamma\left(1+\alpha_n\right)\right) \\
&\le \delta^{-2} \sqrt{2\pi} \left(\Gamma\left(1+2\alpha_n\right) + \Gamma\left(1+\alpha_n\right)\right) \int_0^1 du\, u^{-2\alpha_n} && (2.4.59) \\
&\le 2\sqrt{2\pi} \delta^{-2} \left(1-2\alpha_n\right)^{-1}, && (2.4.60)
\end{aligned}$$

for n large enough such that $\alpha_n < \frac{1}{2}$. This is linear in α_n, and therefore sufficient to show that (2.4.56) tends to zero as first $n \to \infty$ and then $\delta \to 0$.

It remains to bound (I). We use (2.3.11) from Proposition 2.10. As in (2.3.28) we notice that

$$(I) \le a_n/2^n \sum_{k=1}^{2^n-1} \left(F_n\left(\tfrac{k}{2^n}\right) - \mathbb{E}\left[F_n\left(\tfrac{k}{2^n}\right)\right]\right) + 2. \tag{2.4.61}$$

We are looking for an array of positive real numbers, $t_{n,k}$ such that there exists $\eta > 1$ such that we have

$$\mathbb{P}\left(a_n\left(F_n\left(\tfrac{k}{2^n}\right) - \mathbb{E}\left[F_n\left(\tfrac{k}{2^n}\right)\right]\right) > t_{n,k}\right] \leq 2^{-n}n^{-\eta}. \tag{2.4.62}$$

Let $n \in \mathbb{N}$, $k \leq 2^n - 1$. We use Bennett's inequality for $a = 2$. We choose $\bar{b}^2_{n,k}$ by similar bounds as in (2.4.60) for n large enough. Namely, we take

$$\begin{aligned}
&\textstyle\sum_{x\in\mathcal{V}_n} \mathbb{V}\mathrm{ar}\left(\exp\left(-\delta^{1/\alpha_n}k/(2^n\gamma_n(1))\right)\right) \\
\leq\ & 2^n\mathbb{E}\left[\exp\left(-\delta^{1/\alpha_n}k/(2^n\gamma_n(1))\right)\right] \\
\leq\ & 2^n\mathbb{E}\left[\exp\left(-\delta^{1/\alpha_n}k/(2^n\gamma_n(1))\right)\right] \\
\leq\ & 2^n\sqrt{2\pi}/(a_n\delta)\left(2^n/(2k)\right)^{-\alpha_n}\left(\Gamma(1+\alpha_n) + \Gamma(1+2\alpha_n)\right) \\
\leq\ & 2^{n+1}\sqrt{2\pi}/(\delta a_n)\left(2^n/k\right)^{\alpha_n} \qquad\qquad = \bar{b}^2_{n,k}\ ,
\end{aligned} \tag{2.4.63}$$

where we used $\alpha_n < 1/2$ to bound $2^{-\alpha_n}\left(\Gamma(1+\alpha_n) + \Gamma(1+2\alpha_n)\right) \leq 2$. Thus, in order to apply (2.3.11) of Proposition 2.10 for $t = 2^n/a_n t_{n,k}$, the $t_{n,k}$'s have to satisfy the following two conditions

$$t_{n,k} \leq \sqrt{2\pi}/(2\delta)\left(2^n/k\right)^{\alpha_n} \tag{2.4.64}$$

$$t^2_{n,k} \geq a_n 8\sqrt{2\pi}/(2^n\delta)\left(2^n/k\right)^{\alpha_n}\left(n\log 2 + \eta\log n\right)\ . \tag{2.4.65}$$

Setting $t_{n,k}$ to equal the right hand side of (2.4.65), (2.4.64) is satisfied if and only if

$$\begin{aligned}
& 8\sqrt{2\pi}a_n/(2^n\delta)\left(2^n/k\right)^{\alpha_n}\left(n\log 2 + \eta\log n\right) \leq 2\pi/(4\delta^2)\left(2^n/k\right)^{2\alpha_n} \\
\Leftrightarrow\ & 32\delta a_n/(2^n\sqrt{2\pi})\left(n\log 2 + \eta\log n\right) \leq \left(2^n/k\right)^{\alpha_n}\ ,
\end{aligned} \tag{2.4.66}$$

which by 2.4.32 is for n large enough satisfied. Moreover, 2.4.32 implies that there exist $\eta > 1$ and an array of positive real numbers $t_{n,k}$ which are as desired. We conclude by Borel-Cantelli Lemma that, $\mathbb{P}$-a.s.

$$\textstyle\sum_{k=1}^{2^n-1} a_n\left(F_n(k/2^n) - \mathbb{E}\left[F_n(k/2^n)\right]\right) \leq \sum_{k=1}^{2^n-1} t_{n,k}. \tag{2.4.67}$$

Thus, we have for n large enough

$$\begin{aligned}
(I) &\leq \textstyle\sum_{k=1}^{2^n-1} 2^{-n}t_{n,k} + 2 \\
&= \textstyle\sum_{k=1}^{2^n-1} 2^{-(n+1/2)}\left(8a_n\sqrt{2\pi}\delta^{-1}\left(n\log 2 + \eta\log n\right)\right)^{1/2}\left(2^n/k\right)^{\alpha_n/2} + 2 \\
&= 2^{n(\alpha_n/2-3/2)}\left(8\sqrt{2\pi}a_n\delta^{-1}\left(n\log 2 + \eta\log n\right)\right)^{1/2}\textstyle\sum_{k=1}^{2^n-1} k^{-\alpha_n/2} + 2 \\
&\leq 2^{n(\alpha_n/2-3/2)}\left(8\sqrt{2\pi}a_n\delta^{-1}\left(n\log 2 + \eta\log n\right)\right)^{1/2}\textstyle\int_0^{2^n} du\ u^{-\alpha_n/2} + 2 \\
&\leq \left(8\sqrt{2\pi}\left(\delta\left(1 - \tfrac{\alpha_n}{2}\right)\right)^{-1}a_n/2^n\left(n\log 2 + \eta\log n\right)\right)^{1/2} + 2.
\end{aligned} \tag{2.4.68}$$

By 2.4.32 it follows that for n large enough

$$(I) \leq \left(\delta\left(1 - \tfrac{\alpha_n}{2}\right)\right)^{-1/2} + 2. \tag{2.4.69}$$

Thus, we get that, $\mathbb{P}$-a.s., for n large enough,

$$(2.4.57) \leq 3 + 2\sqrt{2\pi}/(\delta(1-\alpha_n)) + (\delta\left(1 - \frac{\alpha_n}{2}\right))^{-1}, \tag{2.4.70}$$

which finishes the proof of (2.4.56).

2.4.4 Conclusions of the proofs of Theorem 2.6 and Theorem 2.7

In Sections 2.4.1-2.4.3, we showed that Conditions (0) - (3) are $\mathbb{P}$-a.s. satisfied. Therefore, we may apply Theorem 2.2 to find that, $\mathbb{P}$-a.s., $S_n^{\alpha_n} \overset{J_1}{\Longrightarrow} M$, where M is the extremal process with distribution function $F(x) = \exp(-1/x)$, $x > 0$. This finishes the proof of Theorem 2.6.

We can show, using the same arguments as in Section 2.3.1, that Theorem 2.7 follows from Theorem 2.6.

2.5 Application to the p spin SK model

This section is devoted to the proof of Theorem 2.8. We show that the conditions of Theorem 2.3 are satisfied for the particular choices of the sequences a_n, c_n, and α_n.

The following lemma from [36] (Proposition 3.1) tells us how to choose the block length θ_n.

Lemma 2.14. *Let P_{π_n} be the law of the simple random walk on Σ_n started in the uniform distribution. Let $\theta_n = \frac{3\ln 2}{2}n^2$. Then, for any $x, y \in \Sigma_n$, and any $i \geq 0$,*

$$\left| \sum_{k=0}^{1} P_{\pi_n} \left(J_n(\theta_n + i + k) = y, J_n(0) = x \right) - 2\pi_n(x)\pi_n(y) \right| \leq 2^{-3n+1}. \tag{2.5.1}$$

This implies that Condition (1-1) holds true for $\theta_n = \frac{3\ln 2}{2}n^2$.

The proof of Condition (2-1) comes in three parts. We first show that $\mathbb{E}\nu_n^t(u,\infty)$ converges to $t\nu(u,\infty)$. Next we prove that $\mathbb{P}$-almost surely, respectively in $\mathbb{P}$-probability, the limit of $\nu_n^t(u,\infty)$ concentrates for every $u > 0$ and every $t > 0$ around its expectation. Lastly we verify that the second part of (2-1) is satisfied in the same convergence mode with respect to the random environment.

2.5.1 Convergence of $\mathbb{E}\nu_n^t(u,\infty)$.

Proposition 2.15. *For every $u > 0$ and $t > 0$*

$$\lim_{n\to\infty} \mathbb{E}\nu_n^t(u,\infty) = \nu^t(u,\infty) \equiv K_p t u^{-1} . \tag{2.5.2}$$

The proof of Proposition 2.15 centers on the following key proposition

Proposition 2.16. *Let for $t > 0$ and an arbitrary sequence u_n,*

$$\bar{\nu}_n^t(u_n,\infty) = k_n(t)\, \mathcal{P}_{\pi_n}\Big(\max_{i=1,\dots,\theta_n} \lambda_n^{-1}(J_n(i))e_{n,i} > u_n^{1/\alpha_n} c_n \Big) . \tag{2.5.3}$$

Then, for every $u > 0$ and $t > 0$,

$$\lim_{n\to\infty} \mathbb{E}\, \bar{\nu}_n^t(u,\infty) = \nu^t(u,\infty) . \tag{2.5.4}$$

The same holds true when u is replaced by $u_n = u\, \theta_n^{-\alpha_n}$.

Proof of Proposition 2.15. By definition, $\nu_n^t(u,\infty)$ is given by

$$\nu_n^t(u,\infty) = k_n(t)\, \mathcal{P}_{\pi_n}\Big(\sum_{i=1}^{\theta_n} \lambda_n^{-1}(J_n(i))e_{n,i} > u_n^{1/\alpha_n} c_n \Big). \tag{2.5.5}$$

The assertion of Proposition 2.15 is then deduced from Proposition 2.16 using the upper and lower bounds

$$\bar{\nu}_n^t(u,\infty) \leq \nu_n^t(u,\infty) \leq \bar{\nu}_n^t(u\theta_n^{-\alpha_n},\infty) . \tag{2.5.6}$$

□

The proof of Proposition 2.16, which is postponed to the end of this section, relies on three Lemmata. In Lemma 2.17 we show that (2.5.4) holds true if we replace the underlying Gaussian process by a simpler Gaussian process H^1. Lemma 2.18 yields (2.5.4) for the maximum over a properly chosen random subset of indices of H^1. We use Lemma 2.20 to conclude the proof of Proposition 2.16.

We start by introducing a Gaussian process H^1. Let v_n be a sequence of integers, where each member is of order n^ω for $\omega \in \left(c+\frac{1}{2}, 1\right)$. Then, H^1 is a centered Gaussian process defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with covariance structure

$$\Delta^1_{i,j} = \begin{cases} 1 - 2pn^{-1}|i-j|, & \text{if } \lfloor i/v_n \rfloor = \lfloor j/v_n \rfloor, \\ 0, & \text{else.} \end{cases} \tag{2.5.7}$$

For a given process $U = \{U_i,\ i \in \mathbb{N}\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ and an index set I define

$$F_n(u_n, U, I) \equiv \mathbb{P}\left(\max\nolimits_{i\in I} e^{\sqrt{n}\beta_n U_i} > u_n^{1/\alpha_n} c_n\right), \tag{2.5.8}$$

and

$$G_n(u_n, U, I) \equiv \mathcal{P}_{\pi_n}\left(\max\nolimits_{i\in I} e^{\sqrt{n}\beta_n U_i} e_{n,i} > u_n^{1/\alpha_n} c_n \middle| \mathcal{F}^J\right). \tag{2.5.9}$$

Lemma 2.17. *For every $u > 0$ and $t > 0$*

$$\lim_{n\to\infty} k_n(t)\mathbb{E}G_n(u, H^1, [\theta_n]) = \nu^t(u, \infty), \tag{2.5.10}$$

where $[k] \equiv \{1, \ldots, k\}$ for $k \in \mathbb{N}$. The same holds true when u is replaced by $u_n = u\,\theta_n^{-\alpha_n}$.

We prove Proposition 2.16 and Lemmata 2.17, 2.18, and 2.20 for fixed $u > 0$ only. To show that the claims also hold for $u_n = u\theta_n^{-\alpha_n}$, it is a simple rerun of their proofs, using $\theta_n^{-\alpha_n} \to 1$ as $n \to \infty$.

Proof. It is shown in Proposition 2.1 of [19] that, by setting the exponentially distributed random variables to 1 in (2.5.9), we get for every $u > 0$ that

$$\lim_{n\to\infty} a_n v_n^{-1} F_n(u, H^1, [v_n]) = \nu(u, \infty)\,. \tag{2.5.11}$$

Assume for simplicity that θ_n is a multiple of v_n. Note that blocks of H^1 of length v_n are independent and identically distributed. Thus,

$$\begin{aligned} k_n(t)F_n(u, H^1, [\theta_n]) &= k_n(t)\left(1 - \left(1 - F_n(u, H^1, [v_n])\right)^{\theta_n/v_n}\right) \\ &\sim k_n(t)\theta_n v_n^{-1} F_n(u, H^1, [v_n]) \\ &\overset{n\to\infty}{\longrightarrow} \nu^t(u, \infty)\,. \end{aligned} \tag{2.5.12}$$

To show that $k_n(t)\mathbb{E}G_n(u, H^1, [\theta_n])$ also converges to $\nu^t(u, \infty)$ as $n \to \infty$ we use same arguments as in (2.5.12) and prove that $a_n v_n^{-1}\mathbb{E}G_n(u, H^1, [v_n]) \to \nu(u, \infty)$ as $n \to \infty$. Using Fubini we have that

$$\begin{aligned} \frac{a_n}{v_n}\mathbb{E}G_n(u, H^1, [v_n]) &= \frac{a_n}{v_n}\int_{c_n u^{1/\alpha_n}}^\infty dz \int_0^\infty dy \frac{f_{\max_{i\in[v_n]} e_{n,i}}(y)}{y} f_{\max_{i\in[v_n]} e^{\beta_n\sqrt{n}H^1(i)}}\left(\tfrac{z}{y}\right) \\ &= \frac{a_n}{v_n}\int_0^\infty dy f_{\max_{i\in[v_n]} e_{n,i}}(y) F_n(u\,y^{-\alpha_n}, H^1, [v_n])\,, \end{aligned} \tag{2.5.13}$$

where $f_Z(\cdot)$ denotes the density function of Z. Since we want to use computations from the proof of Proposition 2.1 in [19], it is essential that the integration area over y is bounded from below and above. We bound (2.5.13) from above by

$$\begin{aligned}
(2.5.13) \quad \leq \quad & a_n v_n^{-1} \mathcal{P}\Big(\max_{i=1,\dots,v_n} e_{n,i} \leq e^{-nv_n^{-1-\delta}} \Big) && (2.5.14)\\
+ \quad & a_n v_n^{-1} \int_{e^{-nv_n^{-1-\delta}}}^{e^{nv_n^{-1/2-\delta}}} dy f_{\max_{i\in[v_n]} e_{n,i}}(y) F_n(u\, t^{-\alpha_n}, H^1, [v_n]) && (2.5.15)\\
+ \quad & a_n v_n^{-1} \mathcal{P}\Big(\max_{i=1,\dots,v_n} e_{n,i} > e^{nv_n^{-1/2-\delta}} \Big) , && (2.5.16)
\end{aligned}$$

where $\delta > 0$ is chosen in such a way that $nv_n^{-1-\delta}$ diverges and $v_n^\delta \gamma_n^2 \to 0$ as $n \to \infty$, i.e. $\delta < \min\left\{2c, \frac{1-\omega}{\omega}\right\}$. Then,

$$(2.5.14) = a_n v_n^{-1} \Big(1 - \exp\Big(-e^{-nv_n^{-1-\delta}}\Big)\Big)^{v_n} \leq a_n e^{-nv_n^{-\delta}} = o\left(e^{-nv_n^{-\delta}(1-\gamma_n^2 v_n^\delta)}\right) , \quad (2.5.17)$$

i.e. (2.5.14) vanishes as $n \to \infty$. Similarly,

$$(2.5.16) = a_n v_n^{-1} \Big(1 - \Big(1 - \exp\Big(-e^{-nv_n^{-1/2-\delta}}\Big)\Big)^{v_n}\Big) = o\Big(e^{\gamma_n^2 n - e^{nv_n^{-1/2-\delta}}}\Big) \overset{n\to\infty}{\longrightarrow} 0 . \quad (2.5.18)$$

As in equation (2.31) in [19] we see that (2.5.15) is given by

$$\int_{e^{-nv_n^{-1-\delta}}}^{e^{nv_n^{-1/2-\delta}}} dy \frac{f_{\max_{i\in[v_n]} e_{n,i}}(y)}{\gamma_n^2 v_n} \sum_{k=1}^{v_n} \int_{D_k''} da_2 \cdots da_{v_n} \int_{\log(uy^{-\alpha_n})}^{\infty} da_1 \frac{e^{-h_k(a_1,\dots,a_{v_n})}}{(2\pi)^{\frac{v_n-1}{2}}} , \quad (2.5.19)$$

where for $k \in \{1, \dots, v_n\}$

$$h_k(a_1, \dots, a_{v_n}) = a_1 - \tfrac{a_1^2 C_1}{\gamma_n^2 n} - \tfrac{1}{2} \textstyle\sum_{i=2}^{v_n} a_i^2 + \tfrac{(a_2+\dots+a_k-a_{k+1}-\dots-a_{v_n})a_1 C_2}{\gamma_n n} , \quad (2.5.20)$$

for some constants $C_1, C_2 > 0$ and a sequence of sets $D_k'' \subseteq \mathbb{R}^{v_n-1}$ such that

$$\gamma_n^{-2} v_n^{-1} \sum_{k=1}^{v_n} \int_{D_k''} da_2 \cdots da_{v_n} (2\pi)^{-v_n/2-1/2} e^{-\frac{1}{2}\sum_{i=2}^{v_n} a_i^2} \overset{n\to\infty}{\longrightarrow} K_p . \quad (2.5.21)$$

The aim is to separate a_1 from $a_2, \dots, a_{v_n}$ in (2.5.20). We bound the mixed terms in e^{-h_k} up to an exponentially small error by 1. This can be done using a large deviation argument for $|a_2 + \dots + a_{v_n}|$ together with the fact that $|\log y| \in \left[nv_n^{-1-\delta}, nv_n^{-1/2-\delta}\right]$. Computations yield that, up to a multiplicative error that tends to 1 as $n \to \infty$ exponentially fast, (2.5.15) is bounded from above by

$$\int_{e^{-nv_n^{-1-\delta}}}^{\infty} dy f_{\max_{i\in[v_n]} e_{n,i}}(y) y^{\alpha_n}\, u^{-1} K_p \leq \nu(u,\infty) \int_0^\infty dy f_{\max_{i\in[v_n]} e_{n,i}}(y) y^{\alpha_n} . \quad (2.5.22)$$

Moreover by Jensen's inequality,

$$\begin{aligned}
(2.5.19) \quad \leq \quad & \nu(u,\infty)\Big(\mathcal{E}_{\pi_n} \max_{i\in[v_n]} e_{n,i}\Big)^{\alpha_n}\\
= \quad & \nu(u,\infty)\Big(\int_0^\infty dy\, \mathcal{P}\Big(\max_{i\in[v_n]} e_{n,i} > y\Big)\Big)^{\alpha_n}\\
= \quad & \nu(u,\infty)\Big(\int_0^\infty dy \Big(1 - (1-e^{-y})^{v_n}\Big)\Big)^{\alpha_n}\\
\leq \quad & \nu(u,\infty) v_n^{\alpha_n} , && (2.5.23)
\end{aligned}$$

which, as $n \to \infty$, converges to $\nu(u, \infty)$.

To conclude the proof of (2.5.10), we bound (2.5.13) from below by

$$(2.5.13) \geq \frac{a_n}{v_n} \int_0^\infty dy f_{e_{n,1}}(y) F_n(u\, y^{-\alpha_n}, H^1, [v_n]) \,. \tag{2.5.24}$$

To show that the right hand side of (2.5.24) is greater than or equal to $\nu(u, \infty)$, one proceeds as before. □

In the following we form a random subset of $[\theta_n]$ in such a way that on the one hand, with high probability, it contains the maximum of $e^{\beta_n \sqrt{n} H^1(i)}$ over all $i \in [\theta_n]$. On the other hand it should be a sparse enough subset of $[\theta_n]$ so that we are able to de-correlate the random landscape and deal with the SK model. This dilution idea is taken from [19].

If the maximum of $e^{\beta_n \sqrt{n} H^1(i)}$ crosses the level $c_n u^{1/\alpha_n}$, then it will typically be much higher so that, due to strong correlation, at least γ_n^{-2} of its direct neighbors will be above the same level. To see this, we consider Laplace transforms. Set for $v > 0$

$$\widehat{F}_n(v, H^1, \theta_n) \equiv \textstyle\int_0^\infty dz\, e^{-zv} \mathbb{P}\left(\delta_n \sum_{i=1}^{\theta_n} \mathbb{1}_{e^{\beta\sqrt{n}H^1(i)} > c_n u^{1/\alpha_n}} > z\right) \,, \tag{2.5.25}$$

where $\delta_n \in [0, 1]$ for every $n \in \mathbb{N}$. We have that

$$\begin{aligned} \widehat{F}_n(v, H^1, \theta_n) &= \tfrac{1}{v}\left(1 - \mathbb{E}\exp\left(-\delta_n \textstyle\sum_{i=1}^{\theta_n} \mathbb{1}_{e^{\beta_n\sqrt{n}H^1(i)} > c_n u^{1/\alpha_n}}\right)\right) \\ &= \tfrac{1}{v}\left(1 - \left(\mathbb{E}\exp\left(-\delta_n \textstyle\sum_{i=1}^{v_n} \mathbb{1}_{e^{\beta_n\sqrt{n}H^1(i)} > c_n u^{1/\alpha_n}}\right)\right)^{\theta_n/v_n}\right). \end{aligned} \tag{2.5.26}$$

From [19], Proposition 1.3, we deduce that for the choice $\delta_n = \gamma_n^2 \rho_n$, where ρ_n is any diverging sequence of order $O(\log n)$,

$$\lim_{n\to\infty} a_n v_n^{-1}\left(1 - \mathbb{E}\exp\left(-\delta_n \textstyle\sum_{i=1}^{v_n} \mathbb{1}_{e^{\beta_n\sqrt{n}H^1(i)} > c_n u^{1/\alpha_n}}\right)\right) = \nu(u, \infty) \,. \tag{2.5.27}$$

Therefore we have for the same choice of δ_n that

$$k_n(t)\widehat{F}_n(v, H^1, \theta_n) \to t v^{-1} \nu(u, \infty) \,. \tag{2.5.28}$$

From this we conclude that if the maximum is above the level $c_n u^{1/\alpha_n}$ then immediately $O(\gamma_n^{-2})$ are above this level. More precisely, we obtain

Lemma 2.18. *Let ρ_n be as described above. Let $\{\xi_{n,i} : \ i \in \mathbb{N},\ n \in \mathbb{N}\}$ be an array of row-wise independent and identically distributed Bernoulli random variables such that $\mathbb{P}(\xi_{n,i} = 1) = 1 - \mathbb{P}(\xi_{n,i} = 0) = \gamma_n^2 \rho_n$, and such that $\{\xi_{n,i} : i \in \mathbb{N},\ n \in \mathbb{N}\}$ is independent of everything else. Set*

$$\mathcal{I}_k = \{i \in \{1, \ldots, k\} : \ \xi_{n,i} = 1\} \,. \tag{2.5.29}$$

Then, for every $u > 0$ and $t > 0$

$$\lim_{n\to\infty} k_n(t)\mathbb{E}G_n(u, H^1, \mathcal{I}_{\theta_n}) = \nu^t(u, \infty) \,. \tag{2.5.30}$$

The same holds true when u is replaced by $u_n = u\, \theta_n^{-\alpha_n}$.

Proof. It is shown in Lemma 2.3 of [19] that

$$\lim_{n\to\infty} a_n v_n^{-1} F_n(u, H^1, \mathcal{I}_{v_n}) = \nu(u, \infty) \,. \tag{2.5.31}$$

Since the random variables $\xi_{n,i}$ are independent, the claim of Lemma 2.18 is deduced by the same arguments as in (2.5.12). □

To conclude the proof of Proposition 2.16, we use a Gaussian comparison result. The following lemma is an adaptation of Theorem 4.2.1of [48].

Lemma 2.19. *Let H^0 and H^1 be Gaussian processes with mean 0 and covariance matrix $\Delta^0 = (\Delta^0_{ij})$ and $\Delta^1 = (\Delta^1_{ij})$, respectively. Set $\Delta^m \equiv \left(\Delta^m_{ij}\right) = \left(\max\{\Delta^0_{ij}, \Delta^1_{ij}\}\right)$ and $\Delta^h \equiv h\Delta^0 + (1-h)\Delta^1$, for $h \in [0,1]$. Then, for $s \in \mathbb{R}$,*

$$\begin{aligned} & \mathbb{P}(\max_{i\in I} H^0(i) \le s) - \mathbb{P}(\max_{i\in I} H^1(i) \le s) \\ \le \quad & \textstyle\sum_{i,j\in I} (\Delta^0_{ij} - \Delta^1_{ij})^+ \exp\left(-\frac{s^2}{1+\Delta^m_{ij}}\right) \int_0^1 dh (1-(\Delta^h_{ij})^2)^{-\frac{1}{2}} \;, \end{aligned} \tag{2.5.32}$$

where $(x)^+ \equiv \max\{0,x\}$.

We use Lemma 2.19 to prove that

Lemma 2.20. *Let H^0 be given by $H^0(i) = n^{-1/2} H_n(J_n(i))$, $i \in \mathbb{N}$. For every $u > 0$ and $t > 0$*

$$\lim_{n\to\infty} k_n(t) E_{\pi_n} |\mathbb{E} G_n(u, H^0, \theta_n) - \mathbb{E} G_n(u, H^1, \theta_n)| = 0. \tag{2.5.33}$$

The same holds true when u is replaced by $u_n = u\theta_n^{-\alpha}$.

Proof. The proof is in the same spirit as that of Proposition 3.1 in [19]. Together with Lemma 2.18, it is sufficient to show that

$$k_n(t) E_{\pi_n} (\mathbb{E} G_n(u, H^1, [\theta_n]) - \mathbb{E} G_n(u, H^0, [\theta_n]))^+ \to 0 \tag{2.5.34}$$

and

$$k_n(t) E_{\pi_n} |\mathbb{E} G_n(u, H^1, \mathcal{I}_{\theta_n}) - \mathbb{E} G_n(u, H^0, \mathcal{I}_{\theta_n})| \to 0 \;. \tag{2.5.35}$$

We do this by an application of Lemma 2.19. Let $\hat{s}_n$ be given by

$$\hat{s}_n = \tfrac{1}{\sqrt{n}\beta_n} \left(\log c_n + \tfrac{\beta_n}{\gamma_n} \log u - \max_{i\in[\theta_n]} \log e_{n,i}\right) \;. \tag{2.5.36}$$

Then we obtain by Lemma 2.19 that

$$\begin{aligned} & (2.5.34) \\ = \quad & k_n(t) E_{\pi_n} \left(\mathbb{E}\mathcal{E}_{\pi_n}\left[\mathbb{1}_{\max_{i\in[\theta_n]} H^1(i)\le \hat{s}_n} - \mathbb{1}_{\max_{i\in[\theta_n]} H^0(i)\le \hat{s}_n} \mid \mathcal{F}^J\right]\right)^+ \\ \le \quad & k_n(t) E_{\pi_n} \textstyle\sum_{i,j\in[\theta_n]} (\Delta^1_{ij} - \Delta^0_{ij})^+ \mathcal{E}_{\pi_n} e^{-\hat{s}_n^2 (1+\Delta^m_{ij})^{-1}} \int_0^1 dh (1-(\Delta^h_{ij})^2)^{-\frac{1}{2}} \;. \end{aligned} \tag{2.5.37}$$

To remove the exponentially distributed random variables in (2.5.37), let $B_n = \{1 \le \max_{i\in[\theta_n]} e_i \le n\}$. We have for $s_n = (n^{1/2}\beta_n)^{-1}\left(\log c_n + \frac{\beta_n}{\gamma_n}\log u - \log n\right)$ that

$$\mathcal{E}_{\pi_n}\left(\mathbb{1}_{B_n} \exp\left(-\hat{s}_n^2 (1+\Delta^m_{ij})^{-1}\right)\right) \le \exp\left(-s_n^2 (1+\Delta^m_{ij})^{-1}\right) \;. \tag{2.5.38}$$

One can check that $k_n(t)\mathcal{P}(B_n^c) \to 0$. Moreover, by definition of s_n, there exists for every $u > 0$ a constant $C < \infty$ such that for n large enough

$$(2.5.34) \le C k_n(t) E_{\pi_n} \textstyle\sum_{i,j\in[\theta_n]} (\Delta^1_{ij} - \Delta^0_{ij})^+ e^{-\gamma_n^2 n (1+\Delta^m_{ij})^{-1}} \int_0^1 dh (1-(\Delta^h_{ij})^2)^{-\frac{1}{2}} \;. \tag{2.5.39}$$

Likewise we deal with (2.5.35). The terms in (2.5.35) are non-zero if and only if $i,j \in \mathcal{I}_{\theta_n}$. By assumption, the probability of this event is $(\gamma_n^2\rho_n)^2$. Hence, (2.5.35) is bounded above by

$$C k_n(t) (\gamma_n^2\rho_n)^2 E_{\pi_n} \textstyle\sum_{i,j\in[\theta_n]} |\Delta^0_{ij} - \Delta^1_{ij}| e^{-\gamma_n^2 n (1+\Delta^m_{ij})^{-1}} \int_0^1 dh (1-(\Delta^h_{ij})^2)^{-\frac{1}{2}} \;. \tag{2.5.40}$$

We divide the summands in (2.5.39) and (2.5.40) respectively into two parts: pairs of i, j such that $\lfloor i/v_n \rfloor \neq \lfloor j/v_n \rfloor$ and those such that $\lfloor i/v_n \rfloor = \lfloor j/v_n \rfloor$. If $\lfloor i/v_n \rfloor \neq \lfloor j/v_n \rfloor$ then we have by definition of H^1 that $\Delta^1_{ij} = 0$. For i, j such that $\lfloor i/v_n \rfloor = \lfloor j/v_n \rfloor$, we have $\Delta^1_{ij} \leq \Delta^0_{ij}$. In view of this, we get after some computations that

$$\text{(2.5.39)} \leq Ck_n(t)E_{\pi_n}\left[\sum_{\lfloor i/v_n \rfloor \neq \lfloor j/v_n \rfloor}^{\theta_n} (\Delta^0_{ij})^- e^{-\gamma_n^2 n}\right], \tag{2.5.41}$$

and

$$\begin{aligned} \text{(2.5.40)} \quad \leq \quad & Ck_n(t)\gamma_n^4\rho_n^2 E_{\pi_n}\left[\sum_{\lfloor i/v_n \rfloor \neq \lfloor j/v_n \rfloor}^{\theta_n} |\Delta^0_{ij}| e^{-\gamma_n^2 n(1+\Delta^0_{ij})^{-1}}\right. \\ + \quad & \left.\sum_{\lfloor i/v_n \rfloor = \lfloor j/v_n \rfloor}^{\theta_n} |\Delta^0_{ij} - \Delta^1_{ij}| e^{-\gamma^2 n(1+\Delta^0_{ij})^{-1}} (1 - (\Delta^0_{ij})^2)^{-\frac{1}{2}}\right]. \end{aligned} \tag{2.5.42}$$

Since $(\Delta^0_{ij})^- = O(n)$ we know by definition of a_n and θ_n that

$$\text{(2.5.41)} \leq C\theta_n n^{3/2}\alpha_n^{-1} e^{-\frac{1}{2}\gamma_n^2 n}, \tag{2.5.43}$$

which tends to zero as $n \to \infty$. Thus (2.5.34) holds true.

To conclude the proof of (2.5.35) we use Lemma 2.25 from the appendix. We get that (2.5.40) is bounded above by

$$\bar{C}ta_n \sum_{d=0}^{n} e^{-\gamma_n^2 n(1+d)^{-1}} \left(\frac{d^2}{v_n n}\mathbb{1}_{d \leq v_n} + \frac{\exp(\eta\gamma_n^2 \min\{d, n-d\})}{v_n \gamma_n^2}\right), \tag{2.5.44}$$

for some $\bar{C} < \infty$ and $\eta < \infty$. With the same arguments as in the proof of (3.3) in [19], we obtain that (2.5.44) tends to zero as $n \to \infty$. □

Proof of Proposition 2.16. Observe that

$$\left|\mathbb{E}\bar{\nu}_n^t(u, \infty) - \nu^t(u, \infty)\right| = \left|k_n(t)E_{\pi_n}\mathbb{E}G_n(u, H^0, [\theta_n]) - \nu^t(u, \infty)\right|, \tag{2.5.45}$$

which is bounded above by

$$k_n(t)E_{\pi_n}\left|\mathbb{E}G_n(u, H^0, [\theta_n]) - \mathbb{E}G_n(u, H^1, [\theta_n])\right| + \left|k_n(t)\mathbb{E}G_n(u, H^1, [\theta_n]) - \nu^t(u, \infty)\right|. \tag{2.5.46}$$

By Lemma 2.17 and Lemma 2.20, both terms vanish as $n \to \infty$ and Proposition 2.16 follows. □

2.5.2 Concentration of $\nu_n^t(u, \infty)$

To verify the first part of Condition (1-1) we control the fluctuation of $\nu_n^t(u, \infty)$ around its mean.

Proposition 2.21. *For every $u > 0$ and $t > 0$ there exists $C = C(p, t, u) < \infty$, such that*

$$\mathbb{E}\left(\bar{\nu}_n^t(u, \infty) - \mathbb{E}\bar{\nu}_n^t(u, \infty)\right)^2 \leq C\gamma_n^{-2} n^{1-p/2}. \tag{2.5.47}$$

The same holds true when u is replaced by $u_n = u\theta_n^{-\alpha_n}$. In particular, for $p > 5$ and $c \in (0, \frac{1}{2})$ or $p = 5$ and $c < \frac{1}{4}$, the first part of Condition (1-1) holds for every $u > 0$ and $t > 0$, $\mathbb{P}$-a.s.

Proof. Let $\left\{e'_{n,i} : i \in \mathbb{N}, n \in \mathbb{N}\right\}$ and J'_n be independent copies of $\{e_{n,i} : i \in \mathbb{N}, n \in \mathbb{N}\}$ and J_n respectively. Write

$$\begin{aligned} \bar{G}_n(u, H^0, [\theta_n]) \quad &\equiv \quad \mathcal{P}_{\pi_n}\left(\max_{i \in [\theta_n]} e^{\beta_n H_n(J_n(i))} e_{n,i} \leq c_n u^{1/\alpha_n} \middle| \mathcal{F}^J\right) \\ \bar{G}_n(u, H^{0'}, [\theta_n]) \quad &\equiv \quad \mathcal{P}_{\pi'_n}\left(\max_{i \in [\theta_n]} e^{\beta_n H_n(J'_n(i))} e'_{n,i} \leq c_n u^{1/\alpha_n} \middle| \mathcal{F}^{J'}\right). \end{aligned} \tag{2.5.48}$$

Then, as in (3.21) in [28],

$$\begin{aligned}\mathbb{E}\left(\mathcal{E}_{\pi_n}\bar{G}_n(u,H^0,[\theta_n])\right)^2 &= \mathbb{E}\mathcal{E}_{\pi_n}\bar{G}_n(u,H^0,[\theta_n])\mathcal{E}_{\pi'_n}\bar{G}_n(u,H^{0'},[\theta_n]) \\ &= \mathcal{E}_{\pi_n}\mathcal{E}_{\pi'_n}\mathbb{E}\bar{G}_n(u,V^0,[2\theta_n])\,, \qquad (2.5.49)\end{aligned}$$

where V^0 is a Gaussian process defined by

$$V^0(i) = \begin{cases} n^{-1/2}H_n(J_n(i)), & \text{if } 1 \le i \le \theta_n, \\ n^{-1/2}H_n(J'_n(i)), & \text{if } \theta_n+1 \le i \le 2\theta_n\,. \end{cases} \qquad (2.5.50)$$

To further express $\left(\mathbb{E}\mathcal{E}_{\pi_n}\bar{G}_n(u,H^0,[\theta_n])\right)^2$, let V^1 be a centered Gaussian process with covariance matrix

$$\Delta^1_{ij} = \begin{cases} \Delta^0_{ij}, & \text{if } \max\{i,j\} \le \theta_n, \text{ or } \min\{i,j\} \ge \theta_n, \\ 0, & \text{else,} \end{cases} \qquad (2.5.51)$$

where $\Delta^0 = (\Delta^0_{ij})$ denotes the covariance matrix of V^0. Then, as in (3.23) in [28],

$$\left(\mathbb{E}\mathcal{E}_{\pi_n}\bar{G}_n(u,H^0,[\theta_n])\right)^2 = \mathcal{E}_{\pi_n}\mathcal{E}_{\pi'_n}\mathbb{E}\bar{G}_n(u,V^1,[2\theta_n])\,. \qquad (2.5.52)$$

As in the proof of Lemma 2.20 we use Lemma 2.19 to obtain that

$$\begin{aligned}& k_n^2(t)\mathbb{E}\left(\mathcal{E}_{\pi_n}\bar{G}_n(u,H^0,[\theta_n]) - \mathbb{E}\mathcal{E}_{\pi_n}\bar{G}_n(u,H^0,[\theta_n])\right)^2 \\ \le\ & 2k_n^2(t)\sum_{\substack{1\le i\le\theta_n \\ \theta_n+1\le j\le 2\theta_n}} E_{\pi_n}E_{\pi'_n}\Delta^0_{ij}e^{-\gamma_n^2 n(1+\Delta^0_{ij})^{-1}}. \qquad (2.5.53)\end{aligned}$$

It is shown in (3.29) of [28] that

$$E_{\pi_n}E_{\pi'_n}\mathbb{1}_{\Delta^0_{ij}=(\frac{m}{n})^p} = 2^{-n}\binom{n}{(n-m)/2}, \qquad \text{for } m \in \{0,\dots,n\}. \qquad (2.5.54)$$

From this, and with the definition of a_n, we have that

$$\begin{aligned}(2.5.53) &\le 2t^2a_n^2\sum_{m=0}^{n}2^{-n}\binom{n}{(n-m)/2}\left(\frac{m}{n}\right)^p\exp\left(\frac{\gamma_n^2 n}{1+(\frac{m}{n})^p}\right) \\ &\le 2t^2\gamma_n^{-2}\sum_{m=0}^{n}2^{-n}n\binom{n}{(n-m)/2}\left(\frac{m}{n}\right)^p\exp\left(\gamma_n^2 n\frac{(\frac{m}{n})^p}{1+(\frac{m}{n})^p}\right) \\ &= 2t^2\gamma_n^{-2}\sum_{d=0}^{n}2^{-n}n\binom{n}{d}\left(1-\frac{2d}{n}\right)^p\exp\left(\gamma_n^2 n\frac{(1-\frac{2d}{n})^p}{1+(1-\frac{2d}{n})^p}\right) \\ &\le 2t^2\gamma_n^{-2}\sum_{d=0}^{n}n^{1/2}\binom{n}{d}\left(1-\frac{2d}{n}\right)_+^p\exp\left(n\Upsilon_{n,p}\left(\tfrac{d}{n}\right)\right)J_n\left(\tfrac{d}{n}\right)\,, \qquad (2.5.55)\end{aligned}$$

where $\Upsilon_{n,p}(u) = \gamma_n^2 - I(u) - \gamma_n^2(1+|1-2u|^p)^{-1}$ and $J_n(u) = 2^{-n}\binom{n}{\lfloor nu\rfloor}\sqrt{\pi n}e^{nI(u)}$ for $I(u) = u\log u + (1-u)\log(1-u) + \log 2$, for $u \in (0,1)$. Note that (2.5.55) has the same form as (3.28) in [8]. Following the strategy of [8], we show that there exist $\delta, \delta' > 0$ and $c > 0$ such that

$$\Upsilon_{n,p} \le \begin{cases} -c\left(u-\frac{1}{2}\right)^2, & \text{if } u \in (\frac{1}{2}-\delta, \frac{1}{2}+\delta), \\ -\delta', & \text{else.} \end{cases} \qquad (2.5.56)$$

Since $\gamma_n = n^{-c}$ this can be done, independently of p, as in [19] (cf. (3.19) and (3.20)). Finally, together with the calculations from (3.28) in [8] we obtain that

$$\mathbb{E}\left(\bar{\nu}_n^t(u,\infty) - \mathbb{E}\bar{\nu}_n^t(u,\infty)\right)^2 \leq C\gamma_n^{-2} n^{1-p/2}. \tag{2.5.57}$$

The same arguments and calculations are used to prove that (2.5.47) also holds when u is replaced by $u_n = u\theta_n^{-\alpha_n}$. Let $p > 5$ and $c \in (0, \frac{1}{2})$ or $p = 5$ and $c < \frac{1}{4}$. Then, by Borel-Cantelli Lemma, for every $u > 0$ and $t > 0$ there exists a set $\Omega(u,t)$ with $\mathbb{P}(\Omega(u,t)) = 1$ such that on $\Omega(u,t)$, for every $\varepsilon > 0$ and n large enough, we have that $|\bar{\nu}_n^t(u,\infty) - \nu^t(u,\infty)| < \varepsilon$ and $|\bar{\nu}_n^t(u_n,\infty) - \nu^t(u,\infty)| < \varepsilon$. From this we conclude that, on $\Omega(u,t)$ and for n large enough,

$$\nu^t(u,\infty) - \varepsilon \leq \nu_n^t(u,\infty) \leq \nu^t(u_n,\infty) + \varepsilon, \tag{2.5.58}$$

i.e. Condition (1-1) is satisfied, for every $u > 0$ and $t > 0$, $\mathbb{P}$-a.s. □

Proposition 2.22. *Let $p = 2,3,4$ and $c \in (0, \frac{1}{2})$ or $p = 5$ and $c > \frac{1}{4}$. Then, the first part of Condition (1-1) holds in $\mathbb{P}$-probability for every $u > 0$ and $t > 0$.*

Proof. For every $\varepsilon > 0$, we bound $\mathbb{P}\left(|\nu_n^t(u,\infty) - \mathbb{E}(\nu_n^t(u,\infty))| > \varepsilon\right)$ from above by

$$\mathbb{P}\left(|\nu_n^t(u,\infty) - k_n(t)\mathcal{E}_{\pi_n} G_n(u, H^0, \mathcal{I}_{\theta_n})| > \varepsilon/3\right) \tag{2.5.59}$$

$$+ \quad \mathbb{P}\left(k_n(t)|\mathcal{E}_{\pi_n} G_n(u, H^0, \mathcal{I}_{\theta_n}) - \mathbb{E}\mathcal{E}_{\pi_n} G_n(u, H^0, \mathcal{I}_{\theta_n})| > \varepsilon/3\right) \tag{2.5.60}$$

$$+ \quad \mathbb{1}_{\{|\mathbb{E}(\nu_n^t(u,\infty)) - k_n(t)\mathbb{E}\mathcal{E}_{\pi_n} G_n(u,H^0,\mathcal{I}_{\theta_n})| > \varepsilon/3\}}. \tag{2.5.61}$$

Observe that by a first order Chebychev inequality,

$$(2.5.59) \leq |\mathbb{E}\nu_n^t(u,\infty) - k_n(t)\mathbb{E}\mathcal{E}_{\pi_n} G_n(u, H^0, \mathcal{I}_{\theta_n})|. \tag{2.5.62}$$

By Lemmata 2.17, 2.18, and 2.20, (2.5.62) tends to zero as $n \to \infty$. For the same reason, (2.5.61) is equal to zero for large enough n. To bound (2.5.60), we calculate the variance of $k_n(t)\mathcal{E}_{\pi_n} G_n(u, H^0, \mathcal{I}_{\theta_n})$. As in the proof of Proposition 2.21 we use Lemma 2.19, but take into account that there can only be contributions to the left hand side of (2.5.32) if $i, j \in \mathcal{I}_{\theta_n}$. This gives us the additional factor $(\gamma_n^2 \rho_n)^2$ in (2.5.53). Therefore the variance of $k_n(t)\mathcal{E}_{\pi_n} G_n(u, H^0, \mathcal{I}_{\theta_n})$ is bounded above by $C(\gamma_n \rho_n)^2 n^{1-p/2}$ which, for all $p \geq 2$, vanishes as $n \to \infty$. Hence, we have proved Proposition 2.22. □

2.5.3 Second part of Condition (2-1)

We proceed as in Section 3.4 in [28] to verify the second part of Condition (2-1) . With the same notation as in (2.1.12), we define for $u > 0$ and $t > 0$

$$\widetilde{\eta}_n^t(u) \equiv k_n(t) n^{-1} \sum_{x \in \Sigma_n} \left(Q_n^u(x)\right)^2 , \tag{2.5.63}$$

$$\eta_n^t(u) \equiv k_n(t) \sum_{x \in \Sigma_n} \sum_{x' \in \Sigma_n} \mu_n(x,x') Q_n^u(x) Q_n^u(x') , \tag{2.5.64}$$

where $\mu_n(\cdot,\cdot)$ is the uniform distribution on pairs $(x,x') \in \Sigma_n^2$ that are at distance 2 apart, i.e.

$$\mu_n(x,x') = \begin{cases} 2^{-n}\frac{2}{n(n-1)}, & \text{if } \operatorname{dist}(x,x') = 2, \\ 0, & \text{else.} \end{cases} \tag{2.5.65}$$

We prove that the expectations of both (2.5.63) and (2.5.64) tend to zero. First and second order Chebychev inequalities then yield that the second part of Condition (2-1) holds in $\mathbb{P}$-probability, respectively $\mathbb{P}$-a.s.

Lemma 2.23. *For every $u>0$ and $t>0$*

$$\lim_{n\to\infty}\mathbb{E}\widetilde{\eta}_n^t(u)=\lim_{n\to\infty}\mathbb{E}\eta_n^t(u)=0\,. \tag{2.5.66}$$

Proof. We show that $\lim_{n\to\infty}\mathbb{E}\eta_n^t(u)=0$. The assertion for $\widetilde{\eta}_n^t(u)$ is proved similarly. Let

$$\bar{Q}_n^u(x)\equiv\mathcal{P}_x\Big(\sum_{j=1}^{\theta_n}\lambda_n^{-1}(J_n(j))e_{n,j}\le c_n u^{1/\alpha_n}\Big)\,. \tag{2.5.67}$$

Rewrite (2.5.59) in the following way

$$\begin{aligned}
&k_n(t)\sum_{x\in\Sigma_n}\sum_{x'\in\Sigma_n}\mu_n(x,x')\Big(1-\bar{Q}_n^u(x)\Big)\Big(1-\bar{Q}_n^u(x')\Big)\\
=\ &k_n(t)\Big[1-\sum_{(x,x')\in\Sigma_n^2}\mu_n(x,x')\Big(\bar{Q}_n^u(x)+\bar{Q}_n^u(x')-\bar{Q}_n^u(x)\bar{Q}_n^u(x')\Big)\Big]\\
=\ &k_n(t)\Big[1-2\sum_{x\in\Sigma_n}\pi_n(x)\bar{Q}_n^u(x)+\sum_{(x,x')\in\Sigma_n^2}\bar{Q}_n^u(x)\bar{Q}_n^u(x')\Big]\,.
\end{aligned} \tag{2.5.68}$$

To shorten notation, write

$$K_n^u\equiv\mathcal{P}_{\pi_n}\Big(\max_{i\in\{\bar{\theta}_n,\dots,\theta_n\}}e^{\sqrt{n}\beta_n H^0(i)}e_{n,i}>c_n u^{1/\alpha_n}\Big|\,\mathcal{F}^J\Big)=\sum_{x\in\Sigma_n}2^{-n}K_n^u(x), \tag{2.5.69}$$

where $\bar{\theta}_n\equiv 2n\log n$ and

$$K_n^u(x)\equiv\mathcal{P}_x\Big(\max_{i\in\{\bar{\theta}_n,\dots,\theta_n\}}e^{\sqrt{n}\beta_n H^0(i)}e_{n,i}>c_n u^{1/\alpha_n}\Big|\,\mathcal{F}^J\Big). \tag{2.5.70}$$

Using the bound $\bar{Q}_n^u(x)\le\mathcal{E}_x(1-K_n^u(x))\equiv\mathcal{E}_x\bar{K}_n^u(x)$, $x\in\Sigma_n$, and taking expectation with respect to the random environment we obtain that

$$\mathbb{E}\eta_n^t(u)\ \le\ k_n(t)-2\Big(k_n(t)-\mathbb{E}\nu_n^t(u,\infty)\Big) \tag{2.5.71}$$

$$+\ k_n(t)\sum_{(x,x')\in\Sigma_n^2}\mu_n(x,x')\mathbb{E}\Big[\mathcal{E}_x\bar{K}_n^u(x)\mathcal{E}_{x'}\bar{K}_n^u(x')\Big]. \tag{2.5.72}$$

For $\bar{G}_n^u\equiv\mathcal{P}_{\pi_n}\Big(\max_{i\in[\theta_n]}e^{\sqrt{n}\beta_n H^0(i)}e_{n,i}\le c_n u^{1/\alpha_n}\Big)$ observe that

$$(2.5.71)\le k_n(t)-2k_n(t)\mathbb{E}\bar{G}_n^u. \tag{2.5.73}$$

We add and subtract $\mathbb{E}\mathcal{E}_{\pi_n}(1-K_n^u)\equiv\mathbb{E}\mathcal{E}_{\pi_n}\bar{K}_n^u$ as well as

$$\sum_{(x,x')\in\Sigma_n^2}\mu_n(x,x')\mathbb{E}\mathcal{E}_x\bar{K}_n^u(x)\mathcal{E}_{x'}\bar{K}_n^u(x'). \tag{2.5.74}$$

Re-arranging the terms and using the bound from (2.5.73) we see that $\mathbb{E}\eta_n^t(u)$ is bounded from above by

$$2k_n(t)\Big(\mathbb{E}\bar{K}_n^u-\mathbb{E}\bar{G}_n^u\Big) \tag{2.5.75}$$

$$+\ k_n(t)\sum_{x,x'}\mu_n(x,x')\mathbb{E}\mathcal{E}_x K_n^u(x)\mathbb{E}\mathcal{E}_{x'}K_n^u(x') \tag{2.5.76}$$

$$+\ k_n(t)\sum_{x,x'}\mu_n(x,x')\Big(\mathbb{E}\Big[\mathcal{E}_x\bar{K}_n^u(x)\mathcal{E}_{x'}\bar{K}_n^u(x')\Big]-\mathbb{E}\mathcal{E}_x\bar{K}_n^u(x)\mathbb{E}\mathcal{E}_{x'}\bar{K}_n^u(x')\Big)\,. \tag{2.5.77}$$

From Proposition 2.16 we conclude that (2.5.75) and (2.5.76) are of order $O\left(\frac{\log n}{n}\right)$ and $O\left(\theta_n a_n^{-1}\right)$ respectively. To control (2.5.77) we use Lemma 2.19 for the processes V^0 and V^1 as in Proposition 2.21. However, due to the fact that we are looking at the chain after $\bar{\theta}_n$ steps, the comparison is simplified. More precisely, let $\mathcal{A}_n\equiv\Big\{\forall\bar{\theta}_n\le i\le\theta_n:\ \operatorname{dist}(J_n(i),J_n'(i))>n(1-\rho(n))\Big\}\subset$

$\mathcal{F}^J \times \mathcal{F}^{J'}$, where $\rho(n)$ is of the order of $\sqrt{n^{-1}\log n}$. Then, on $\mathcal{A}_n$, by Lemma 2.19 and the estimates from (2.5.35),

$$\mathbb{E}\left[\bar{K}_n^u(x)\bar{K}_n^u(x')\right] - \mathbb{E}\bar{K}_n^u(x)\mathbb{E}\bar{K}_n^u(x') \leq 2\gamma_n^{-2} \sum_{\substack{1\leq i\leq \theta_n \\ \theta_n+1\leq j\leq 2\theta_n}} \Delta_{ij}^0 e^{-\gamma_n^2 n(1+\Delta_{ij}^0)^{-1}} \leq O(\theta_n^2 a_n^{-2}). \tag{2.5.78}$$

Moreover, on $\mathcal{A}_n^c$,

$$\mathbb{E}\left[\bar{K}_n^u(x)\bar{K}_n^u(x')\right] - \mathbb{E}\bar{K}_n^u(x)\mathbb{E}\bar{K}_n^u(x') \leq O(a_n^{-1}). \tag{2.5.79}$$

But in Lemma 3.7 from [28] it is shown that for a specific choice of $\rho(n)$ and every $x \in \Sigma_n$

$$\begin{aligned} P\left(\mathcal{A}_n \middle| \operatorname{dist}(J_n(0), J_n'(0)) = 2\right) &\geq 1 - n^{-8} \\ P_x\left(\mathcal{A}_n^c\right) &\leq n^{-4}. \end{aligned} \tag{2.5.80}$$

Therefore we obtain that $\lim_{n\to\infty} \mathbb{E}\eta_n^t(u) = 0$. □

Remark. Lemma 2.23 immediately implies that the second part of Condition (2-1) holds in $\mathbb{P}$-probability. To show that it is satisfied $\mathbb{P}$-almost surely for $p > 5$ and $c \in (0, \frac{1}{2})$ or $p = 5$ and $c < \frac{1}{4}$ it suffices to control the variance of (2.5.75). We use the same concentration results as in Proposition 2.21 to obtain that the variance of $k_n(t)(\bar{K}_n^u - \bar{G}_n^u)$, which is given by

$$k_n^2(t)\left[\mathbb{E}\left(\bar{K}_n^u - \mathbb{E}\bar{K}_n^u\right)^2 + \mathbb{E}\left(\bar{G}_n^u - \mathbb{E}\bar{G}_n^u\right)^2 - 2\left(\mathbb{E}\bar{G}_n^u\bar{K}_n^u - \mathbb{E}\bar{G}_n^u\mathbb{E}\bar{K}_n^u\right)\right], \tag{2.5.81}$$

is bounded from above by $C\gamma_n^{-2}n^{1-p/2}$.

2.5.4 Condition (3-1)

We show that Condition (3-1) is $\mathbb{P}$- a.s. satisfied for every $\delta > 0$.

Lemma 2.24. *We have, $\mathbb{P}$-a.s.,*

$$\limsup_{n\to\infty}\left(a_n\left(c_n\delta^{1/\alpha_n}\right)^{-1}\mathcal{E}_{\pi_n}\lambda_n^{-1}(J_n(1))e_{n,1}\mathbb{1}_{\lambda_n^{-1}(J_n(1))e_{n,1}\leq c_n\delta^{1/\alpha_n}}\right)^{\alpha_n} < \infty, \quad \forall \delta > 0. \tag{2.5.82}$$

Proof. We begin by proving that for every $\delta > 0$, for n large enough,

$$\begin{aligned} \tfrac{a_n}{c_n\delta^{1/\alpha_n}}\mathcal{E}_{\pi_n}\mathbb{E}\lambda_n^{-1}(J_n(1))e_{n,1}\mathbb{1}_{\lambda_n^{-1}(J_n(1))e_{n,1}\leq c_n\delta^{1/\alpha_n}} &= \textstyle\sum_{x\in\Sigma_n} 2^{-n}\mathbb{E}Y_{n,\delta}(x) \\ &\leq 4(\delta\gamma_n\beta_n)^{-1}, \end{aligned} \tag{2.5.83}$$

where $Y_{n,\delta}(x) \equiv a_n\left(c_n\delta^{1/\alpha_n}\right)^{-1}\lambda_n^{-1}(x)e_{n,1}\mathbb{1}_{\lambda_n^{-1}(x)e_{n,1}\leq c_n\delta^{1/\alpha_n}}$, for $x \in \Sigma_n$.

For $x \in \Sigma_n$ we have that

$$\begin{aligned} \mathbb{E}Y_{n,\delta}(x) &= a_n(c_n\delta^{1/\alpha_n})^{-1}(2\pi)^{-1/2}\int_0^\infty dy\int_{-\infty}^{y_n} dz\; ye^{-y-\frac{z^2}{2}+\beta_n\sqrt{n}z} \\ &= a_n(c_n\delta^{1/\alpha_n})^{-1}(2\pi)^{-1/2}\int_0^\infty dy\int_{\beta_n\sqrt{n}-y_n}^{\infty} dz\; ye^{-y+\frac{\beta_n^2 n}{2}-\frac{z^2}{2}}, \end{aligned} \tag{2.5.84}$$

where $y_n \equiv (\sqrt{n}\beta_n)^{-1}\left(\log c_n + \frac{\beta_n}{\gamma_n}\log\delta - \log y\right)$ for $y > 0$. In order to use estimates on Gaussian integrals, we divide the integration area over y into $y \leq n^2$ and $y > n^2$.

For $y > n^2$, there exists a constant $C' > 0$ such that

$$(2\pi)^{-1/2}a_n(c_n\delta^{1/\alpha_n})^{-1}\int_{n^2}^{\infty}dy\int_{-\infty}^{y_n}dz\; ye^{-y-\frac{z^2}{2}+\beta_n\sqrt{n}z} \le C'a_n n^4 e^{-n^2}, \tag{2.5.85}$$

which vanishes as $n \to \infty$.

Let $y \le n^2$. By definition of c_n we have $\beta_n\sqrt{n} - y_n = \sqrt{n}\beta_n\left(1 - \frac{\gamma_n}{\beta_n} - \frac{\log\delta}{\gamma_n\beta_n n} + \frac{\log y}{\beta_n^2 n}\right)$. Since $\alpha_n \to 0$ as $n \to \infty$, it follows that for n large enough $\beta_n\sqrt{n} - y_n > 0$. But then, since $\mathbb{P}(Z > z) \le (\sqrt{2\pi})^{-1}z^{-1}e^{-z^2/2}$ for any $z > 0$ and Z being a standard Gaussian,

$$\int_0^{n^2}dy\int_{-y_n+\beta_n\sqrt{n}}^{\infty}dz\; ye^{-y+\frac{\beta_n^2 n}{2}-\frac{z^2}{2}} \le \int_0^{n^2}dy\frac{ye^{-y}}{\beta_n\sqrt{n}-y_n}e^{\frac{\beta_n^2 n}{2}-\frac{(\beta_n\sqrt{n}-y_n)^2}{2}}\,. \tag{2.5.86}$$

Plugging in the definition of a_n and c_n, (2.5.85) and (2.5.86) yield that, for n large enough, up to a multiplicative error that tends to 1 as $n \to \infty$ exponentially fast,

$$\begin{aligned}
(2.5.84) \;&\le\; \textstyle\int_0^{n^2}dy\; y^{\alpha_n}e^{-y}(\gamma_n\beta_n\delta)^{-1}\left(1-\frac{\gamma_n}{\beta_n}-\frac{\log\delta}{n\gamma_n\beta_n}+\frac{\log y}{\beta_n^2 n}\right)^{-1}e^{2\log\delta\log n(n\gamma_n\beta_n)^{-1}}\\
&\le\; 2\int_0^{n^2}dy\; y^{\alpha_n}e^{-y}(\gamma_n\beta_n\delta)^{-1}\\
&\le\; 2\Gamma\left(1+\tfrac{\gamma_n}{\beta_n}\right)(\gamma_n\beta_n\delta)^{-1}\,,
\end{aligned} \tag{2.5.87}$$

where $\Gamma(\cdot)$ denotes the gamma function. Since $\Gamma(1+\alpha_n) \le 2$ for $\alpha_n \le 1$, the claim of (2.5.83) holds true for every $\delta > 0$ for n large enough.

Lemma 3.10 from [28] yields that for every $\delta > 0$ there exists $\kappa > 0$ such that

$$\mathbb{E}\left(\mathcal{E}_{\pi_n}Y_{n,\delta}\right)^2 - \left(\mathbb{E}\mathcal{E}_{\pi_n}Y_{n,\delta}\right)^2 \le a_n^2\left(c_n\delta^{1/\alpha_n}\right)^{-2}n^{1-p/2} \le e^{-n^\kappa}, \tag{2.5.88}$$

where $\mathcal{E}_{\pi_n}Y_{n,\delta} \equiv \sum_{x\in\Sigma_n}2^{-n}Y_{n,\delta}(x)$. By Borel-Cantelli Lemma for every $\delta > 0$ there exists a set $\Omega(\delta)$ with $\mathbb{P}(\Omega(\delta)) = 1$ such that on $\Omega(\delta)$, for every $\varepsilon > 0$ there exists $n' \in \mathbb{N}$ such that

$$\mathcal{E}_{\pi_n}Y_{n,\delta} \le 4\left(\gamma_n\beta_n\delta\right)^{-1} + \varepsilon, \quad \forall n \ge n'. \tag{2.5.89}$$

Setting $\Omega^\tau \equiv \bigcap_{\delta\in\mathbb{Q}\cap(0,\infty))}\Omega(\delta)$, we have $\mathbb{P}(\Omega^\tau) = 1$.

Let $\delta > 0$ and $\varepsilon > 0$. We can always find $\delta' \in \mathbb{Q}$ such that $\delta \le \delta' \le 2\delta$. Note that $Y_{n,\delta}$ is increasing in δ. Moreover, by (2.5.89) there exists $n' = n'(\delta',\varepsilon)$ such that on Ω^τ and for $n \ge n'$

$$\left(\mathcal{E}_{\pi_n}Y_{n,\delta}\right)^{\alpha_n} \le \left(\mathcal{E}_{\pi_n}Y_{n,\delta'}\right)^{\alpha_n} \le \left(4\left(\gamma_n\beta_n\delta'\right)^{-1}+\varepsilon\right)^{\alpha_n} \le 4\left(\gamma_n\beta_n\delta'\right)^{-\alpha_n}. \tag{2.5.90}$$

Since $(\gamma_n\beta_n)^{-\alpha_n} \to 1$ as $n \to \infty$, we obtain the assertion of Lemma 2.24. $\square$

2.5.5 Proof of Theorem 2.8

We are now ready to conclude the proof of Theorem 2.8.

First let $p > 5$ and $\gamma_n = n^{-c}$ for $c \in \left(0, \frac{1}{2}\right)$, or $p = 5$ and $c > \frac{1}{4}$. Then we know by Propositions 2.16 and 2.21 that for every $u > 0$ there exists a set $\Omega(u)$ with $\mathbb{P}(\Omega(u)) = 1$ and such that on $\Omega(u)$

$$\lim_{n\to\infty}\nu_n^t(u,\infty) = K_p t u^{-1}, \quad \forall t > 0. \tag{2.5.91}$$

The mapping that maps u to $\nu_n^t(u,\infty)$ is decreasing on $(0,\infty)$ and its limit, u^{-1}, is continuous on the same interval. Therefore, setting $\Omega_1^\tau = \bigcap_{u\in(0,\infty)\cap\mathbb{Q}}\Omega(u)$, we have $\mathbb{P}(\Omega_1^\tau) = 1$ and (2.5.91) holds true for every $u > 0$ on Ω_1^τ. By Section 2.5.3 there also exists a subset Ω_2^τ with full measure and such that the second part of Condition (2-1) holds on Ω_2^τ.

Condition (3-1) holds $\mathbb{P}$-a.s. by Lemma 2.24. Finally, we are left with the verification of Condition (0) for the invariant measure $\pi_n(x) = 2^{-n}$, $x \in \Sigma_n$. For $v > 0$, we have that

$$\sum_{x\in\Sigma_n} 2^{-n} e^{-v^{\alpha_n} c_n \lambda_n(x)} = \sum_{x\in\Sigma_n} 2^{-n} \mathcal{P}_{\pi_n}\left(\lambda_n^{-1}(x) e_{n,1} > c_n v^{\alpha_n}\right). \tag{2.5.92}$$

By similar calculations as in (2.5.87), we see that, for n large enough and $x \in \Sigma_n$,

$$\mathbb{E}\mathcal{P}_{\pi_n}\left(\lambda_n^{-1}(x) e_{n,1} > c_n v^{\alpha_n}\right) \sim a_n^{-1}\gamma_n^2 v^{-1}, \tag{2.5.93}$$

which tends to zero as $n \to \infty$. By a first order Chebychev inequality we conclude that for every $v > 0$ Condition (0) is satisfied $\mathbb{P}$-a.s. As before, by monotonicity and continuity, this implies that Condition (0) holds $\mathbb{P}$-a.s. for every $v > 0$. Therefore, Theorem 2.8 holds in this case.

For $p = 2, 3, 4$ and $c \in \left(0, \frac{1}{2}\right)$ or $p = 5$ and $c \geq \frac{1}{4}$, we know from Propositions 2.16, 2.21, and Section 2.5.3 that Condition (2-1) is satisfied in $\mathbb{P}$-probability, whereas Condition (0) and (3-1) hold $\mathbb{P}$-a.s. This concludes the proof of Theorem 2.8.

2.5.6 Proof of Theorem 2.9

We use Theorem 2.8 to prove the claim of Theorem 2.9.

By the same arguments as in the proof of Theorem 1.5 in [28], we obtain that for $t > 0$, $s > 0$, and $\varepsilon \in (0, 1)$ the correlation function $\mathcal{C}_n^\varepsilon(t, s)$ can, with very high probability and $\mathbb{P}$- a.s., be approximated by

$$\begin{aligned} \mathcal{C}_n^\varepsilon(t,s) &= (1-o(1))\,\mathcal{P}_{\pi_n}(\mathcal{R}_n \cap (t^{\alpha_n}, (t+s)^{\alpha_n}) = \emptyset) \\ &= (1-o(1))\,\mathcal{P}_{\pi_n}(\mathcal{R}_{\alpha_n} \cap (t, t+s) = \emptyset), \end{aligned} \tag{2.5.94}$$

where $\mathcal{R}_n$ is the range of the blocked clock process S_n^b and $\mathcal{R}_{\alpha_n}$ is the range of $\left(S_n^b\right)^{\alpha_n}$. By Theorem 2.8 we know that $\left(S_n^b\right)^{\alpha_n} \overset{J_1}{\Longrightarrow} M_\nu$, $\mathbb{P}$-a.s. for $p > 5$ if $c \in (0, \frac{1}{2})$, $p = 5$ if $c < \frac{1}{4}$, and in $\mathbb{P}$-probability else. By Proposition 4.8 in [56] we know that the range of M_ν is the range of a Poisson random measure ξ' with intensity measure $\nu'(u, \infty) = \log u - \log K_p$. Thus, writing $\mathcal{R}_M$ for the range of M_ν, we get that

$$\mathcal{P}(\mathcal{R}_M \cap (t, t+s) = \emptyset) = \mathcal{P}(\xi'(t, t+s) = 0) = e^{-\nu'(t,t+s)} = \tfrac{t}{t+s}. \tag{2.5.95}$$

The claim of Theorem 2.9 follows.

2.6 Appendix

In the appendix we state and prove a lemma that is needed in the proof of Lemma 2.20.

Lemma 2.25. *Let $D_{ij} = \operatorname{dist}(J_n(i), J_n(j))$ and $\Delta_d^0 = (1 - 2dn^{-1})^p$. For any $\eta > 0$ there exists a constant $\bar{C} < \infty$ such that for n large enough and $d \in \{0, \dots, n\}$*

$$k_n(t) \textstyle\sum_{\lfloor i/v_n\rfloor = \lfloor j/v_n\rfloor}^{\theta_n} E_{\pi_n} \mathbb{1}_{D_{ij}=d} |\Delta_d^0 - \Delta_{ij}^1| \leq \bar{C} t a_n \frac{d^2}{v_n n} \mathbb{1}_{d\leq v_n}, \tag{2.6.1}$$

$$k_n(t) \textstyle\sum_{\lfloor i/v_n\rfloor \neq \lfloor j/v_n\rfloor}^{\theta_n} E_{\pi_n} \mathbb{1}_{D_{ij}=d} \leq \bar{C} t \frac{a_n \exp\left(\eta\gamma_n^2 \min\{d, n-d\}\right)}{v_n \gamma_n^2}. \tag{2.6.2}$$

Proof. We use ideas from Section 3 in [8] and Section 4 in [19] and write the distance process $D_{ij} = \mathrm{dist}(J_n(i), J_n(j))$ as the Ehrenfest chain $Q_n = \{Q_n(k) : k \in \mathbb{N}\}$, which is a birth-death process with state space $\{0, \ldots, n\}$ and transition probabilities $p_{k,k-1} = 1 - p_{k,k+1} = \frac{k}{n}$ for $k \in \{0, \ldots, n\}$. Denote by P_k the law and E_k the expectation of Q_n starting in k. Let moreover $T_d = \inf\{k \in \mathbb{N} : Q_n(k) = d\}$. By the Markov property of J_n, we have under P_0, in distribution, that

$$\mathrm{dist}(J_n(0), J_n(k)) \stackrel{d}{=} \mathrm{dist}(J_n(j), J_n(j+k)) \stackrel{d}{=} Q_n(k) , \quad \forall j, k \geq 0. \tag{2.6.3}$$

Recall for the proof of (2.6.1) that if $\lfloor i/v_n \rfloor = \lfloor j/v_n \rfloor$, we have that $\Delta^1_{ij} \leq \Delta^0_{i,j}$. Moreover, since for such i, j necessarily $|i-j| \leq v_n$ we have that $D_{ij} \leq v_n$. Thus, let $d \in \{1, \ldots, v_n\}$. By Lemma 4.2 in [8] we deduce that there exists a constant $C < \infty$, independent of d, such that

$$k_n(t) \textstyle\sum_{\lfloor i/v_n \rfloor = \lfloor j/v_n \rfloor}^{\theta_n} E_{\pi_n} \mathbb{1}_{D_{ij}=d} \leq C t a_n \,. \tag{2.6.4}$$

Moreover,

$$\left(\Delta^0_d - \Delta^1_{ij}\right) = \left(1 - \tfrac{2d}{n}\right)^p - \left(1 - \tfrac{2p|i-j|}{n}\right) = \tfrac{2p}{n}\left(|i-j| - d\right) + O\left(\tfrac{d^2}{n^2}\right). \tag{2.6.5}$$

Therefore the main contributions in (2.6.1) are of the form

$$\begin{aligned} \textstyle\sum_{\lfloor i/v_n \rfloor = \lfloor j/v_n \rfloor}^{\theta_n} \left(|i-j| - d\right) E_{\pi_n} \mathbb{1}_{D_{ij}=d} &= v_n \textstyle\sum_{i=1}^{\lfloor \theta_n/v_n \rfloor} \sum_{j=i+1}^{i+v_n} (j - i - d)\, E_{\pi_n} \mathbb{1}_{D_{ij}=d} \\ &= v_n \textstyle\sum_{i=1}^{\lfloor \theta_n/v_n \rfloor} \sum_{j=1}^{v_n} E_0 \mathbb{1}_{Q_n(j)=d}\, (j-d). \end{aligned} \tag{2.6.6}$$

Setting $Z \equiv \sum_{j=1}^{v_n} \mathbb{1}_{Q_n(j)=d}\,(j-d)$, (2.6.6) is nothing but $\theta_n E_0 Z$. It is shown in [19] (page 31-32) that there exists a constant $C < \infty$, independent of d, such that

$$\begin{aligned} E_0 Z &\leq C E_0 \left(T_d - d\right) \mathbb{1}_{T_d < v_n} \\ &\leq C \left(E_0 T_d - d P_0 \left(T_d < v_n\right)\right) \leq C \left(E_0 T_d - d\left(1 - v_n^{-1} E_0 T_d\right)\right) , \end{aligned} \tag{2.6.7}$$

where the last inequality is obtained by a first order Chebychev inequality. To calculate $E_0 T_d$ we use the following classical formulas (see e.g. [49], Chapter 2.5)

$$E_0 T_d = \textstyle\sum_{l=1}^d E_{l-1} T_l, \quad \text{where} \tag{2.6.8}$$

$$E_{l-1} T_l = \tfrac{1}{p_{l,l-1}} \textstyle\prod_{i=1}^l \tfrac{p_{i,i-1}}{p_{i-1,i}} \left(1 + \sum_{j=1}^{l-1} \prod_{k=1}^j \tfrac{p_{k,k-1}}{p_{k-1,k}}\right). \tag{2.6.9}$$

Plugging in the transition probabilities we obtain that for all $l \leq d$,

$$\begin{aligned} E_{l-1} T_l &= \tfrac{n}{l} \left(\textstyle\prod_{i=1}^l \tfrac{i}{n-i+1} + \sum_{j=1}^{l-1} \prod_{k=j+1}^l \tfrac{k}{n-k+1}\right) \\ &= \tfrac{n}{l} \textstyle\sum_{j=0}^{l-1} \prod_{k=j+1}^l \tfrac{k}{n-k+1}. \end{aligned} \tag{2.6.10}$$

For any $l \leq d$ and $0 \leq j \leq l-1$ we have that

$$\tfrac{n}{l} \textstyle\prod_{k=j+1}^l \tfrac{k}{n-k+1} \leq \tfrac{n}{d} \prod_{k=j+1}^l \tfrac{d}{n-d} \,. \tag{2.6.11}$$

In view of (2.6.8) we get that

$$E_0 T_d \leq \textstyle\sum_{l=1}^d \tfrac{1}{1-2dn^{-1}} \left(1 - \left(\tfrac{d}{n-d}\right)^l\right) \leq \tfrac{d}{(1-2dn^{-1})}. \tag{2.6.12}$$

But then, since $\frac{d}{n} \to 0$ as $n \to \infty$ and $d \leq v_n$, there exists a constant $C' < \infty$, independent of d, such that

$$E_0 Z \leq C' \tfrac{d^2}{v_n}. \tag{2.6.13}$$

Together with (2.6.4) and (2.6.5) this concludes the proof of (2.6.1).

For the proof of (2.6.2) we distinguish several cases. If $\|d\| \equiv \min\{d, n-d\} > (\log n)^{1+\varepsilon}\gamma_n^{-2}$ for some fixed $\varepsilon > 0$ then the claim of (2.6.2) is deduced from the bound

$$k_n(t) \sum_{\lfloor i/v_n \rfloor \neq \lfloor j/v_n \rfloor}^{\theta_n} E_{\pi_n} \mathbb{1}_{D_{ij}=d} \leq a_n t\theta_n \ll a_n t \frac{e^{\eta\|d\|\gamma_n^2}}{v_n\gamma_n^2}. \tag{2.6.14}$$

Assume next that $\|d\| \leq (\log n)^{1+\varepsilon}\gamma_n^{-2}$. It is shown in [19], (page 35-36), that in this case one can neglect values of d such that $d \geq \frac{n}{2}$. Thus, let $d \leq (\log n)^{1+\varepsilon}\gamma_n^{-2}$. Note that

$$k_n(t) \sum_{\lfloor i/v_n \rfloor \neq \lfloor j/v_n \rfloor}^{\theta_n} E_{\pi_n} \mathbb{1}_{D_{ij}=d} \leq k_n(t) \sum_{k=0}^{\theta_n} \sum_{m=j_k}^{\theta_n} E_{\pi_n} \mathbb{1}_{D_{k,k+m}=d}\ , \tag{2.6.15}$$

where $j_k = \inf\{i \in \mathbb{N} :\ \lfloor k/v_n \rfloor \neq \lfloor (k+i)/v_n \rfloor\}$.

We further distinguish the cases $j_k \leq 2d$ and $j_k > 2d$. If $j_k \leq 2d$ then, setting $Z_{j_k}(d) \equiv \sum_{m=j_k}^{\theta_n} \mathbb{1}_{D_{k,k+m}=d}$, we have $Z_{j_k}(d) \leq Z_0(d)$. It is shown on page 685 in [8] that there exists $C < \infty$, independent of d, such that $E_0 Z_0(d) \leq C$. Since moreover $|\{k \in \{1, \ldots, \theta_n\} :\ j_k \leq 2d\}| \leq 2\frac{d\theta_n}{v_n}$, we know that for every $\eta > 0$ there exists $C' < \infty$ such that

$$k_n(t) \sum_{k=0}^{\theta_n} \sum_{m=j_k}^{\theta_n} E_{\pi_n} \mathbb{1}_{D_{ij}=d} \leq Ct\frac{a_n d}{v_n} \leq C' t \frac{a_n e^{\eta\gamma_n^2\|d\|}}{v_n\gamma_n^2}. \tag{2.6.16}$$

Let $j_k > 2d$, i.e. in particular $Z_{j_k}(d) \leq Z_{2d}(d)$. By the Markov property and by Lemma 4.2 in [8] we obtain that there exists $C < \infty$ such that

$$E_0 Z_{2d}(d) \leq P_0(T_d \in (2d, \theta_n)) \left(1 + E_d\left(\sum_{k=1}^{\theta_n} \mathbb{1}_{Q_n(k)=d}\right)\right) \leq CP_0(T_d \in (2d, \theta_n)). \tag{2.6.17}$$

The probability that Q gets from 0 to d after $2d$ steps is bounded by the probability that it takes at least d steps to the left, i.e.

$$P_0(T_d \in (2d, \theta_n)) \leq \binom{2d}{d}\left(\frac{d}{n}\right)^d \leq 2d\left(\frac{4d}{n}\right)^d \ll \frac{d}{v_n}. \tag{2.6.18}$$

The claim follows as in (2.6.16). This concludes the proof of (2.6.2). $\square$

Chapter 3

Convergence of clock processes on infinite graphs and aging in Bouchaud's asymmetric trap model on $\mathbb{Z}^d$

Véronique Gayrard and Adéla Švejda

Using a method developed by Durrett and Resnick [33] we establish general criteria for the convergence of properly rescaled clock processes of random dynamics in random environments on infinite graphs. This extends the results of [37], [28], and [29], and gives a unified development of most of the convergence theorems for clock processes. As a first application we prove that Bouchaud's asymmetric trap model on $\mathbb{Z}^d$ exhibits a normal aging behavior for all $d \geq 2$. Namely, we show that certain two-time correlation functions, among which the classical probability to find the process at the same site at two time points, converge, as the age of the process diverges, to the distribution function of the arcsine law. As a byproduct we prove that the fractional kinetics process ages.

3.1 Introduction and main results

This introduction is made of three parts. In the first we describe the general setting and formulate the problems of interest. We state our abstract results in Section 3.1.2. Section 3.1.3 contains the application to Bouchaud's asymmetric trap model.

3.1.1 Markov jump processes in random environments and clock processes

Let $G = (\mathcal{V}, \mathcal{L})$ be a loop-free graph. The random environment is a collection of random variables, $\{\tau(x), x \in \mathcal{V}\}$, defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, that are only assumed to be positive. On $\mathcal{V}$ we consider a continuous time Markov jump process, X, with initial distribution μ, whose jump rates $(\lambda(x,y))_{x,y\in\mathcal{V}}$ satisfy

$$\tau(x)\lambda(x,y) = \tau(y)\lambda(y,x), \quad \forall (x,y) \in \mathcal{L},\ x \neq y. \tag{3.1.1}$$

This implies that X is reversible with respect to the random measure on $\mathcal{V}$ that assigns to $x \in \mathcal{V}$ the mass $\tau(x)$.

Clock processes of X have recently been at the center of attention in connection with the study of aging and/or anomalous diffusions. Relevant questions on both topics can be formulated by writing X as a time change of another Markov process J,

$$X(t) = J(S^{\leftarrow}(t)), \qquad t \geq 0, \tag{3.1.2}$$

and making judicious choices of S, the so-called *clock process*. Here $S^{\leftarrow}$ denotes the generalized right continuous inverse of S. When studying aging the focus usually is on the total time elapsed along trajectories of X of a given length. This is given by the *discrete time clock process*

$$S(k) \equiv \sum_{i=0}^{k-1} \lambda^{-1}(J(i))e_i, \qquad k \geq 1, \tag{3.1.3}$$

where J is the discrete time chain with transition probabilities

$$p(x,y) \equiv \lambda(x,y)/\lambda(x) \quad \text{if } (x,y) \in \mathcal{L}, \tag{3.1.4}$$

and zero else,

$$\lambda(x) \equiv \textstyle\sum_{y:(x,y)\in\mathcal{L}} \lambda(x,y), \quad x \in \mathcal{V}, \tag{3.1.5}$$

is the inverse of the mean holding time of X at x, and $\{e_i,\ i = 0,1,2,\ldots\}$ is an independent collection of i.i.d. mean one exponential random variables. Knowledge of the large k behavior of S combined with relation (3.1.2) then allows to deduce information on the long time behavior of the two-time correlation functions that are used to quantify aging in theoretical physics. When interested in scaling limits one looks at (3.1.2) from a different angle. One aims at expressing the process X as a time change of another continuous time process, J, for which the usual functional limit theorem holds. One is then naturally led to study the *continuous time clock process*

$$S(t) \equiv \int_0^t \lambda^{-1}(J(s))\widetilde{\lambda}(J(s))ds, \qquad t \geq 0, \tag{3.1.6}$$

where $\widetilde{\lambda}(x)$ denotes the inverse of the mean holding time of J at x.

It emerged from the bulk of works carried out in the past decade that the occurrence of stable subordinators as the limit of properly rescaled clock processes provides a basic mechanism for both aging and anomalous diffusive behaviors to set in in two main types of models. The first are phenomenological models – the so-called trap models of Bouchaud *et al.* [24, 26, 58, 59]. Introduced in theoretical physics to account for the phenomenon of aging then newly discovered in the physics of spin glasses, these are simple Markov jump processes that describe the dynamics of spin glasses on long time scales in terms of activated barrier crossing in landscapes made of random 'traps'. Another class of models stems from looking at the actual dynamics of microscopic spin glasses. Interesting such dynamics are Glauber dynamics on state spaces $\mathcal{V}_n = \{-1,1\}^n$ reversible with respect to the Gibbs measures associated to random Hamiltonians of mean-field spin glasses, such as the REM and p-spin SK models.

The first connection between microscopic dynamics of spin systems and trap models was made in [11], [13], [9] for a variant of the Glauber dynamics of the REM (the random hopping dynamics, hereafter RHD) on time scales close to equilibrium, and extended in [17] to shorter time scales (but still exponential in n). There it is shown that the properly rescaled discrete time clock process (3.1.3) converges $\mathbb{P}$-a.s. to a stable subordinator. These results were partially extended to the p-spin SK models in [8], for all $p \geq 3$ and in a range of exponentially long time scales, whereas it was shown in [19] that on sub-exponential times scales the clock process no longer converges to a stable subordinator but to an extremal process, and this for all $p \geq 2$; both these results were obtained in $\mathbb{P}$-law only.

The field gained new momentum with the paper [37]. Based on a method developed by Durrett and Resnick [33] in the late 70's to prove functional limit theorems for dependent random variables, a fresh view on the convergence of clock processes in random environment was proposed and general criteria for convergence of clock processes to subordinators were given. This allowed to improve all earlier results on aging of the RHD of the REM [36] and p-spin SK models [28], [29], yielding $\mathbb{P}$-a.s. results for all $p > 4$ (in $\mathbb{P}$-probability else), and paved the way for new advances [39]. In all the papers mentioned above clock processes are used to control suitable time-time correlation functions, and aging is deduced.

Meanwhile, in a different line of research, an important class of trap models on $\mathbb{Z}^d$ known as Bouchaud's asymmetric trap model (hereafter BATM) [58, 59] was fully investigated both from the view point of aging and scaling limits, in different dimensions and for different values of the asymmetry parameter $\theta \in [0,1]$ (see Section 3.1.3 for the definition of BATM). In what follows we call BTM the 'symmetric' version of the model, obtained by setting $\theta = 0$. Aging was first proved in the seminal paper [35] for BTM on $\mathbb{Z}$, and extended to BATM on $\mathbb{Z}$ in [14]. Emphasis was first given to the discrete clock process of BTM in [18], for $d = 2$, and later in [16], for $d \geq 2$. In both these papers it is proved that for suitable scalings, the clock process converges to a stable subordinator. This is used in [18] to study aging via correlation functions, and in [16] to prove convergence of the properly normalized BTM to the so-called Fractional-Kinetics process (see (3.1.48)). More recently, [34] established aging for transient variants of BTM on $\mathbb{Z}^d$ for all $d \geq 1$. The continuous time clock process (3.1.6) came into play later, in the study of BATM on $\mathbb{Z}^d$, $d \geq 2$, [2], [6], [31], [53]. There, J is chosen as the so-called variable speed random walk (hereafter VSRW), that is to say, the continuous time Markov chain with rates $\widetilde{\lambda}(x,y) = \tau(x)\lambda(x,y)$. This is a central object in the literature on random conductance models and its scaling limit is well-understood (for the most recent and strongest results see [4] and [1]). Convergence of the rescaled clock process to a stable subordinator is established in [2], [31], [53] under various assumptions on d and using various techniques (see Section 3.1.3 for a detailed discussion). Consequences for the scaling limit of BATM are drawn but not, to our knowledge, for correlation functions.

The question naturally arises as to whether the method put forward in [37] could allow to make progress on this issue. How to implement it however is not straightforward. The formulation of the general, abstract criteria for convergence of clock processes of [37] and [28] was geared to the setting of sequences of finite graphs suited for dealing with mean field spin glasses. Furthermore, in all applications, explicit use is made of the fact that the discrete time chain J in (3.1.3) admits an invariant probability measure and is, moreover, sufficiently fast mixing. In contrast, the arena of BATM on $\mathbb{Z}^d$ is that of dynamics on infinite graphs that do not admit of an invariant probability measure.

In the present paper we address this question in the general setting of Markov jump processes on infinite graphs that satisfy (3.1.1). We formulate abstract sufficient conditions for properly rescaled clock processes of the form (3.1.2) (both continuous or discrete) to converge to stable subordinators. (It will be seen that the rôle of the invariant measure is now played by a certain 'mean empirical measure'.) We then apply this result to two classes of models: First, we study BATM for all $d \geq 2$ and our method enables us to control several (classical or natural) correlation functions through which the aging behavior of the process can be characterized, and prove the existence of 'normal aging'. Then, we study a version of BTM, for $d \geq 2$, in which the process jumps from one site to another according to the distribution of a discrete time Markov chain that whose distribution does not depend on the random environment but is more general than that of the simple random walk. In particular, the class of dynamics we consider does not necessarily perform nearest neighbor jumps. This way, we improve some of the results obtained in [34]. More precisely, we prove almost sure aging behavior for a sub-class of the models considered in [34], where the statements hold in law, respectively in probability.

3.1.2 Main results

In this paper, we consider continuous and discrete time clock processes in a unified setting and introduce notations that allow to handle them simultaneously. From now on let J be either a continuous or discrete time Markov chain having transition probabilities (3.1.4) and initial distribution μ. Continuous time chains are assumed to be non-explosive (see Chapter 3.5 in [55]). To a Markov chain J we associate a process $\ell = \{\ell_t(x), x \in \mathcal{V}, t \geq 0\}$ and a sequence $\widetilde{\Lambda} = \{\widetilde{\lambda}(x), x \in \mathcal{V}\}$ defined as follows. When J is a continuous time Markov chain $\widetilde{\lambda}(x)$ is the holding time parameter of J at x and $\ell_t(x)$ is the local time

$$\ell_t(x) \equiv \textstyle\int_0^t \mathbb{1}_{J(s)=x}\, ds, \tag{3.1.7}$$

namely, the total time spent by J at x in the time interval $[0,t]$. When J is a discrete time Markov chain we set $\widetilde{\lambda}(x) \equiv 1$. In this case $\ell_t(x)$ is defined through

$$\ell_t(x) \equiv \textstyle\sum_{i=0}^{\lfloor t\rfloor-1} e_i \mathbb{1}_{J(i)=x}, \tag{3.1.8}$$

where $\{e_i, i = 0,1,2,\dots\}$ is a collection of i.i.d. mean one exponential random variables independent of everything else. Observe that this is the local time of a continuous time Markov chain whose mean holding times are identically one. The clock process is then given by

$$S^J(t) \equiv \sum_{x\in\mathcal{V}} \ell_t(x)\widetilde{\lambda}(x)\lambda^{-1}(x), \quad t \geq 0. \tag{3.1.9}$$

Notice that this definition is consistent with (3.1.3) and (3.1.6). In particular, one can check that the relation (3.1.2) between S^J, J, and X is satisfied. In the sequel we write P_μ for the law of J and $\mathcal{P}_\mu$ for the law of X with initial distribution μ. We also write $P_x \equiv P_{\delta_x}$ and $\mathcal{P}_x \equiv \mathcal{P}_{\delta_x}$. Of course these are random measures on $(\Omega, \mathcal{F}, \mathbb{P})$.

Let a_n and c_n be non-decreasing sequences. We think of c_n as the time scale on which the process X is observed, and of a_n as an auxiliary time scale for the Markov chain J. The question of interest now becomes to find conditions for the re-scaled sequence

$$S_n^J(t) \equiv c_n^{-1} \sum_{x\in\mathcal{V}} \widetilde{\lambda}(x)\lambda^{-1}(x)\ell_{\lfloor a_n t\rfloor}(x), \quad t \geq 0, \tag{3.1.10}$$

to converge weakly, as a sequence of random elements in the space $D[0,\infty)$ of càdlàg functions on $[0,\infty)$, $\mathbb{P}$-almost surely in the random environment.

To answer this question we use a method developed by Durrett and Resnick [33] that yields criteria for sums of correlated random variables to converge that are particularly useful when applied to clock processes. Following [28], we will not apply it to S_n^J directly but rather to a 'blocked' version of S_n^J. Namely, we introduce a new sequence, θ_n, chosen such that $\theta_n \ll a_n$, and use it to define the block variables

$$Z_{n,k}^J \equiv c_n^{-1} \sum_{x\in\mathcal{V}} \widetilde{\lambda}(x)\lambda^{-1}(x)\left(\ell_{\theta_n k}(x) - \ell_{\theta_n(k-1)}(x)\right), \quad n \geq 1,\, k \geq 1. \tag{3.1.11}$$

The reason for this is that in many examples of interest, we do know that the jumps of the limiting clock process do not come from isolated jumps of S_n^J but from block variables, either because of strong local spatial correlations of the random environment (as in spin glasses) or because of strong local temporal correlations of the process J (as e.g. in BTM on $\mathbb{Z}^2$) or by a conjunction of these reasons. Set $k_n(t) \equiv \lfloor \lfloor a_n t\rfloor/\theta_n\rfloor$. The blocked clock process $S_n^{J,b}$ is then defined by

$$S_n^{J,b}(t) \equiv \sum_{k=0}^{k_n(t)-1} Z_{n,k+1}^J, \quad t \geq 0. \tag{3.1.12}$$

The convergence criteria we obtain bear on a small number of quantities that we now introduce. For $x \in \mathcal{V}$ and $u > 0$ let

$$Q_n^u(x) \equiv \mathcal{P}_x\left(Z_{n,1}^J > u\right) \tag{3.1.13}$$

be the tail distribution of $Z_{n,1}^J$, starting in x. For each fixed $t > 0$, we construct a probability measure on $\mathcal{V}$ through

$$\pi_n^t(x) \equiv E_\mu\Big((k_n(t))^{-1} \sum_{k=1}^{k_n(t)-1} \mathbb{1}_{J(k\theta_n)=x}\Big), \quad x \in \mathcal{V}. \tag{3.1.14}$$

This is the empirical measure induced by the sequence $\{J(k\theta_n), k = 1, \dots k_n(t) - 1\}$, averaged over P_μ. Note that Q_n^u and π_n^t are not random in the chain J. Using these quantities, we define

$$\nu_n^t(u,\infty) \equiv k_n(t) \sum_{x\in\mathcal{V}} \pi_n^t(x) Q_n^u(x), \tag{3.1.15}$$

and

$$\sigma_n^t(u,\infty) \equiv k_n(t) \sum_{x\in\mathcal{V}} \pi_n^t(x)(Q_n^u(x))^2. \tag{3.1.16}$$

We are now ready to introduce the conditions of our main theorem. They are stated for given sequences a_n, c_n, θ_n, a given initial distribution μ, and fixed $\omega \in \Omega$.

(A-0) For all $u > 0$

$$\lim_{n\to\infty} \mathcal{P}_\mu\left(S_n^{J,b}(0) > u\right) = 0. \tag{3.1.17}$$

(A-1) For all $t > 0$ there exists $c < \infty$ such that, uniformly in $x \in \mathcal{V}$,

$$\lim_{n\to\infty} \sum_{k=1}^{k_n(t)-1} P_\mu(J(k\theta_n) = x) = 0, \tag{3.1.18}$$

and

$$\lim_{n\to\infty} \sum_{k=1}^{k_n(t)-1} P_x(J(k\theta_n) = x) < c. \tag{3.1.19}$$

(A-2) There exists a sigma-finite measure ν on $(0,\infty)$ satisfying $\int_0^\infty (1 \wedge x) d\nu(x) < \infty$ such that for all $t > 0$ and $u > 0$ such that $\nu(\{u\}) = 0$ and $\nu(u,\infty) < \infty$,

$$\lim_{n\to\infty} \nu_n^t(u,\infty) = t\nu(u,\infty). \tag{3.1.20}$$

(A-3) For all $t > 0$ and all $u > 0$ such that $\nu(\{u\}) = 0$ and $\nu(u,\infty) < \infty$,

$$\lim_{n\to\infty} \sigma_n^t(u,\infty) = 0. \tag{3.1.21}$$

(A-4) For all $t > 0$,

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} k_n(t) \sum_{x\in\mathcal{V}} \pi_n^t(x) \mathcal{E}_x Z_{n,1}^J \mathbb{1}_{\{Z_{n,1}^J \le \varepsilon\}} = 0. \tag{3.1.22}$$

Theorem 3.1. *Assume that there exist sequences a_n, c_n, and θ_n and an initial distribution μ such that Conditions (A-0)-(A-4) are satisfied $\mathbb{P}$-a.s. Then, $\mathbb{P}$-a.s., as $n \to \infty$,*

$$S_n^{J,b} \Rightarrow V_\nu, \tag{3.1.23}$$

where V_ν is a subordinator with Lévy measure ν and zero drift. Convergence holds weakly in the space $D[0,\infty)$ equipped with Skorohod's J_1 topology.

Let us emphasize that our statement is made for $S_n^{J,b}$ and holds in the strong J_1 topology, which immediately implies that S_n^J converges to the same limit in the weaker M_1 topology. As we just explained (see discussion below (3.1.11)), in many models of interest it is not true that S_n^J converges in the J_1 topology, but more information than contained in M_1 statements can be obtained by introducing a blocked clock process, $S_n^{J,b}$. (This is the case in the p-spin SK models [8], [28], and BTM on $\mathbb{Z}^2$ [18].) Of course in the case of continuous time clock processes, forming blocks is needed in order to make sense of writing convergence to subordinators statements in the J_1 topology. Let us finally stress that it is crucial for applications to correlation functions to make statements that are valid in the J_1 topology.

Let us comment on Conditions (A-0)-(A-4). Condition (A-0) is a condition on the initial distribution and ensures that the initial increment $S_n^{J,b}(0)$ converges to zero as $n \to \infty$. Conditions (A-2)-(A-4) have the same form as Conditions (A2-1)-(A3-1) in [28] where sequences of finite state reversible Markov jump processes are studied. There it is assumed that J_n admits a unique invariant measure, π_n, and θ_n is chosen large compared to the mixing time of J_n (see Condition (A1-1)). In the present setting, the empirical measure averaged over P_μ replaces the measure π_n, and Condition (A-1) plays the same rôle as Condition (A1-1). More precisely these conditions allow to replace J dependent, respectively J_n dependent, quantities by their average over P_μ, respectively P_{π_n}. We conclude this discussion with a lemma that sheds light on the complementarity of Theorem 3.1 and Theorem 1.3 in [28]; indeed the former can only be satisfied by J's that are transient, respectively null-recurrent, whereas the latter is designed for positive recurrent J's.

Lemma 3.2. *Let $x \in \mathcal{V}$. If x is transient then* (3.1.18) *and* (3.1.19) *are satisfied for any $\theta_n \gg 1$, whereas if x is positive recurrent they cannot be satisfied. In particular, (A-1) cannot hold if J admits an invariant probability measure.*

When J is random in the random environment the conditions of Theorem 3.1 may not be easy to handle. We now present an additional condition, (B-5), that enables us to replace π_n^t in (A-2)-(A-4) by a deterministic probability measure $\overline{\pi}_n^t$. In this way, all the dependence on the random environment in (A-2)-(A-4) is confined to the Q_n^u's. The following conditions, stated for given sequences a_n, c_n, θ_n, a given initial distribution μ, and for fixed $\omega \in \Omega$, imply the conditions of Theorem 3.1.

(B-5) Set $\mathcal{A}_n = \{(x,k) : \ x \in \mathcal{V}, \ k \in [k_n(t)-1]\}$, where $[m] \equiv \{0,\dots,m\}$. There exists a sequence of functions $h_n : \mathcal{V} \to [0,1]$ such that for all $t > 0$ and all $n \in \mathbb{N}$, the set $\mathcal{A}_n$ can be decomposed into the disjoint union of two sets, $\mathcal{A}_n^1$ and $\mathcal{A}_n^2$, satisfying

$$\lim_{n\to\infty} \sup_{(x,k)\in\mathcal{A}_n^1} \frac{|P_\mu(J(k\theta_n)=x)) - h_{k\theta_n}(x)|}{h_{k\theta_n}(x)} = 0, \tag{3.1.24}$$

and

$$\lim_{n\to\infty} \sum_{(x,k)\in\mathcal{A}_n^2} |P_\mu(J(k\theta_n)=x) - h_{k\theta_n}(x)| Q_n^u(x) = 0, \tag{3.1.25}$$

$$\lim_{n\to\infty} \sum_{(x,k)\in\mathcal{A}_n^2} |P_\mu(J(k\theta_n)=x) - h_{k\theta_n}(x)| \mathcal{E}_x Z_{n,1}^J \mathbb{1}_{\{Z_{n,1}^J \le \varepsilon\}} = 0. \tag{3.1.26}$$

Observe that proving (3.1.24) corresponds to proving a uniform local central limit theorem for J.

For each $t > 0$ we define the measure $\overline{\pi}_n^t$, using h_n, through

$$\overline{\pi}_n^t(x) = (k_n(t))^{-1} \sum_{k=1}^{k_n(t)-1} h_{k\theta_n}(x), \quad x \in \mathcal{V}. \tag{3.1.27}$$

By analogy to (3.1.15) and (3.1.16) we set for $t > 0$, $u > 0$

$$\overline{\nu}_n^t(u,\infty) \equiv k_n(t) \sum_{x\in\mathcal{V}} \overline{\pi}_n^t(x) Q_n^u(x), \tag{3.1.28}$$

$$\overline{\sigma}_n^t(u,\infty) \equiv k_n(t) \sum_{x\in\mathcal{V}} \overline{\pi}_n^t(x) (Q_n^u(x))^2. \tag{3.1.29}$$

The next conditions are nothing but Conditions (A-2)-(A-4) with π_n^t replaced by $\overline{\pi}_n^t$.

(B-2) There exists a sigma-finite measure ν on $(0,\infty)$ satisfying $\int_0^\infty (1\wedge x) d\nu(x) < \infty$ such that for all $t > 0$ and $u > 0$ such that $\nu(\{u\}) = 0$ and $\nu(u,\infty) < \infty$,

$$\lim_{n\to\infty} \overline{\nu}_n^t(u,\infty) = t\nu(u,\infty). \tag{3.1.30}$$

(B-3) For all $t > 0$ and all $u > 0$ such that $\nu(\{u\}) = 0$ and $\nu(u,\infty) < \infty$,

$$\lim_{n\to\infty} \overline{\sigma}_n^t(u,\infty) = 0. \tag{3.1.31}$$

(B-4) For all $t > 0$,

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} k_n(t) \sum_{x\in\mathcal{V}} \overline{\pi}_n^t(x) \mathcal{E}_x Z_{n,1}^J \mathbb{1}_{\{Z_{n,1}^J \le \varepsilon\}} = 0. \tag{3.1.32}$$

Theorem 3.3. *Assume that there exist sequences a_n, c_n, and θ_n and an initial distribution μ such that Conditions (A-0), (A-1), (B-2)-(B-5) are satisfied $\mathbb{P}$-a.s. Then, $\mathbb{P}$-a.s.,*

$$S_n^{J,b} \Rightarrow V_\nu, \tag{3.1.33}$$

where V_ν is a subordinator with Lévy measure ν and zero drift. Convergence holds weakly in the space $D[0,\infty)$ equipped with Skorohod's J_1 topology.

3.1.3 Application to Bouchaud's asymmetric trap model (BATM)

We now use Theorem 3.3 to prove aging in Bouchaud's asymmetric trap model on $\mathbb{Z}^d$. Here $\mathcal{V} = \mathbb{Z}^d$, $d \ge 2$, $\mathcal{L}$ is the set of nearest neighbors on $\mathbb{Z}^d$, and $\mu \equiv \delta_0$. The random environment, $\left\{\tau(x), x\in\mathbb{Z}^d\right\}$, is a collection of i.i.d. random variables, with tail distribution given by

$$\mathbb{P}(\tau(0) > u) = \begin{cases} Cu^{-\alpha}(1+L(u)), & u\in(\bar{c},\infty), \\ 1, & u\in(0,\bar{c}], \end{cases} \tag{3.1.34}$$

where $\alpha \in (0,1)$, $\bar{c}, C \in (0,\infty)$ are constants, and $L : (0,\infty) \to \mathbb{R}$ is a function that obeys $L(u) \to 0$ as $u \to \infty$. We write $x \sim y$ if x, y are nearest neighbors in $\mathbb{Z}^d$. The jump rates of X depend on a parameter, $\theta \in [0,1]$, and are given by

$$\lambda(x,y) = (\tau(x))^{-1}(\tau(x)\tau(y))^\theta, \quad \text{if } x\sim y, \tag{3.1.35}$$

and zero else. Consider now the VSRW of this model, namely, the continuous time Markov chain, $\widetilde{J}$, with jump rates

$$\widetilde{\lambda}(x,y) = (\tau(x)\tau(y))^\theta, \quad \text{if } x\sim y, \tag{3.1.36}$$

and zero else. Our interest is in the continuous time clock process $S^{\widetilde{J}}$ defined (as in (3.1.10)) through

$$S^{\widetilde{J}}(t) = \textstyle\sum_{x\in\mathcal{V}} \ell_t(x)\widetilde{\lambda}(x)\lambda^{-1}(x) = \sum_{x\in\mathcal{V}} \ell_t(x)\tau(x). \tag{3.1.37}$$

Our first theorem states convergence of the blocked clock process, $S_n^{\widetilde{J},b}$, for appropriate choices of block lengths, θ_n, in J_1 topology.

Theorem 3.4. *Let* $c_n = n$ *and take*

$$\theta_n = n^{\alpha\gamma_2}\mathbb{1}_{d=2} + (\log n)^{\gamma_3}\mathbb{1}_{d\geq 3}, \tag{3.1.38}$$
$$a_n = n^{\alpha}(\log n)^{1-\alpha}\mathbb{1}_{d=2} + n^{\alpha}\mathbb{1}_{d\geq 3}, \tag{3.1.39}$$

where $\gamma_2 \in (0, 1/6), \gamma_3 > 9$. *Then,* $\mathbb{P}$*-a.s., as* $n \to \infty$,

$$S_n^{\widetilde{J},b} \Rightarrow V_\alpha, \tag{3.1.40}$$

where V_α *is a subordinator with Lévy measure* $\nu(u,\infty) = \mathcal{K}u^{-\alpha}$, *for* $\mathcal{K} = \mathcal{K}(d,\alpha,\theta) > 0$, *and zero drift. Convergence holds weakly in the space* $D[0,\infty)$ *equipped with Skorohod's* J_1 *topology.*

In all papers where the clock process $S_n^{\widetilde{J}}$ was studied before the main objective was to prove scaling limits for BATM. It was first proved in [2] that the properly rescaled process converges to a fractional kinetics process for $d \geq 3$. This was extended to $d = 2$ in [31]. Shortly after [2], [53] gave an alternative proof of this result for $d \geq 5$. The method of [2, 31] relies on blocking with block length $\theta_n = \varepsilon n^\alpha$. In contrast, [53] proposed a method of proof that does not use blocking. Both approaches resulted in M_1 convergence for $S_n^{\widetilde{J}}$. (Note that because $S^{\widetilde{J}}$ is a continuous time clock process, the method of [53] does not allow to obtain J_1 convergence statements for $S^{\widetilde{J}}$.) As already mentioned this is not enough to control correlation functions.

Let us comment on our choices of θ_n. Because $\widetilde{J}$ is recurrent when $d = 2$ and transient otherwise two cases must be distinguished. When $d = 2$ we first remark that (A-1) would be satisfied for any $\theta_n \gg \log n$. There our constraint on θ_n comes from (A-2)-(A-4). In the course of verifying these conditions one sees that θ_n must be chosen in such a way that the mean values of local times during $[0,\theta_n]$ are of the order of $\log n$. Since these mean values are of order $\log\theta_n$ we take $\theta_n = n^{\alpha\gamma_2}$. When $d \geq 3$ Conditions (A-1)-(A-4) can a priori be verified for any diverging θ_n. Here the constraint (3.1.38) on θ_n comes from using precise heat kernel estimates for $\widetilde{J}$, taken from [4], which are only valid for large enough time intervals (of course this was already the case in $d = 2$).

We now present our results on aging. Theorem 3.4 allows to control several correlation functions, which we now introduce. The first is the classical correlation function

$$\mathcal{C}_s^1(1,\rho) \equiv \mathcal{P}\left(X(s) = X(s(1+\rho))\right), \quad s > 0,\ \rho > 0, \tag{3.1.41}$$

which is the probability that at the beginning and the end of the time interval $(s, s(1+\rho))$ the process is in the same site. The second correlation function is the probability that during a certain time interval the process stays inside a ball of a certain radius. Specifically, writing $\theta_s \equiv \theta_{\lfloor s\rfloor}$,

$$\mathcal{C}_s^2(1,\rho) \equiv \mathcal{P}\left(\max\nolimits_{v\in(s,s(1+\rho))}|X(s) - X(v)| \leq (\theta_s\log\theta_s)^{1/2}\right),\ s > 0,\ \rho > 0. \tag{3.1.42}$$

Notice that $\mathcal{C}_s^1$ and $\mathcal{C}_s^2$ clearly contain different information. Our third and last correlation function combines them both. For $s > 0$, $\rho > 0$ we define

$$\mathcal{C}_s^3(1,\rho) \equiv \mathcal{P}\Big(X(s) = X(s(1+\rho)), \max_{v\in(s,s(1+\rho))}|X(s) - X(v)| \leq (\theta_s\log\theta_s)^{1/2}\Big). \tag{3.1.43}$$

The proof of the next theorem relies on a well-known scheme, that first appeared in [15], that links aging to the arcsine law for subordinators through the convergence of the clock process $S_n^{\widetilde{J},b}$. Let Asl_α denote the distribution function of the generalized arcsine law,

$$\mathrm{Asl}_\alpha(u) \equiv \tfrac{\sin\alpha\pi}{\pi}\textstyle\int_0^u(1-x)^{-\alpha}x^{\alpha-1}dx, \quad u \in [0,1]. \tag{3.1.44}$$

Theorem 3.5. *Let $d \geq 2$. Under the assumptions of Theorem 3.4, for $i = 1, 2, 3$, $\mathbb{P}$-a.s.,*

$$\lim_{s\to\infty} \mathcal{C}_s^i(1, \rho) = \mathrm{Asl}_\alpha(1/(1 + \rho)), \quad \rho > 0. \tag{3.1.45}$$

As pointed out below Theorem 3.4, it was proved that the rescaled process

$$X_s(t) \equiv a_s^{-1/2} X(st), \quad t \geq 0, \tag{3.1.46}$$

converges to the fractional kinetics process. Observe that the radius of the balls in (3.1.42) for which Theorem 3.5 holds is very small compared to the normalization of X_s, $(\theta_s \log\theta_s)^{1/2} \ll a_s^{1/2}$. From this and Theorem 3.5 one readily deduces that the correlation function defined, for $\varepsilon > 0$, by

$$\mathcal{C}_s^\varepsilon(1, \rho) \equiv \mathcal{P}\left(\max_{v\in(1,1+\rho)} |X_s(1) - X_s(v)| \leq \varepsilon\right), \quad s > 0, \rho > 0, \tag{3.1.47}$$

converges to the arcsine distribution function. Interestingly, this, in turns, enables us to deduce results on the aging behavior of the fractional kinetics process itself. This is the content of Theorem 3.6 below. Recall that the fractional kinetics process is defined by

$$Z_{d,\alpha}(t) \equiv B_d(V_\alpha^{\leftarrow}(t)), \quad t \geq 0, \tag{3.1.48}$$

where B_d is a standard Brownian motion on $\mathbb{R}^d$ started in 0, V_α is an α-stable subordinator with zero drift that is independent of B_d, and $V_\alpha^{\leftarrow}(t) = \inf\{v : V_\alpha(v) > t\}$ its generalized right-continuous inverse. By analogy to (3.1.47) define

$$\mathcal{C}^\varepsilon(1, \rho) \equiv \mathcal{P}\left(\max_{v\in(1,1+\rho)} |Z_{d,\alpha}(1) - Z_{d,\alpha}(v)| \leq \varepsilon\right), \quad s > 0, \rho > 0. \tag{3.1.49}$$

Theorem 3.6. *Let $d \geq 2$. Under the assumptions of Theorem 3.4, $\mathbb{P}$-a.s.,*

$$\lim_{\varepsilon\to 0}\lim_{s\to\infty} \mathcal{C}_s^\varepsilon(1, \rho) = \lim_{\varepsilon\to 0} \mathcal{C}^\varepsilon(1, \rho) = \mathrm{Asl}_\alpha(1/(1 + \rho)), \quad \rho > 0. \tag{3.1.50}$$

Remark. As a final remark notice that our results are only valid for $d \geq 2$. It is known that the situation in $d = 1$ is completely different, see [35], [14]. The clock process converges to the integral of the local time of a Brownian motion on $\mathbb{R}$ with respect to the so-called (random) speed measure which is the scaling limit of the random environment and the scaling limit is a singular diffusion on $\mathbb{R}$; see e.g. [16] and [31] for further discussions.

3.1.4 Application to dynamics of trap models

Let us now study class of dynamics of trap models that is not random in the random environment. The set of vertices is again $\mathbb{Z}^d$ and the random environment is the same as in Section 3.1.3. The set of edges and the jump distribution, which is deterministic in the random environment but more general than simple random walk, are constructed as follows. Let $\{Y_i, i \geq 1\}$ be a sequence of i.i.d. $\mathbb{Z}^d$-valued random variables with distribution π_Y. We assume that π_Y is symmetric, has finite and non-singular covariance matrix Γ, and that there exists $\delta > 0$ such that $Ee^{t\cdot Y_1} < \infty$ for all $|t| \leq \delta$. The set of edges, $\mathcal{L}$, is implicitly defined through

$$(x, y) \in \mathcal{L} \quad \Leftrightarrow \quad \pi_Y(x - y) > 0. \tag{3.1.51}$$

The Markov chain J is constructed as follows. We set $J(0) = 0$, $0 \in \mathbb{Z}^d$, and define

$$J(k) = J(0) + \textstyle\sum_{i=1}^{k-1} Y_i, \quad k \geq 1. \tag{3.1.52}$$

In other words, J has initial distribution δ_0 and transition matrix P with entries

$$p(x,y) = \pi_Y(x-y). \tag{3.1.53}$$

Our process of interest, X, is, for $t \geq 0$ given by $X(t) = J(S^{\leftarrow}(t))$, where S is the discrete time clock process,

$$S^J(k) \equiv \textstyle\sum_{i=0}^{k-1} \tau(J(i))e_i, \quad k \geq 1, \tag{3.1.54}$$

where $\{e_i : i \in \mathbb{N}\}$ is a collection of i.i.d. exponential random variables with mean 1. Hence, the jump rates of X are given by $\lambda(x,y) = \pi_Y(x-y)/\tau(x)$, for $x, y \in \mathcal{V}$. Therefore, for (3.1.1) to hold, it is necessary that

$$\pi_Y(x-y) = \lambda(x,y)\tau(x) = \lambda(y,x)\tau(y) = \pi_Y(y-x), \quad \forall (x,y) \in \mathcal{L}, \tag{3.1.55}$$

that is to say the symmetry condition for π_Y is necessary. The other assumptions on π_Y are of technical nature. (see the paragraph before (3.5.8) and , respectively Section 3.6.5). More precisely, under the above assumptions we have Gaussian upper bounds for $P(J(n) = x)$ (see (3.5.14)) and control on the exit times of certain balls (see (3.5.8) and Section 3.6.5).

We use Theorem 3.1 to prove the following convergence result for a blocked version of the clock process S.

Theorem 3.7. *Let $c_n = n$ and take*

$$\begin{aligned} \theta_n &= n^{\alpha\gamma_2}\mathbb{1}_{d=2} + (\log n)^{\gamma_3}\mathbb{1}_{d\geq 3}, & (3.1.56)\\ a_n &= n^{\alpha}(\log n)^{1-\alpha}\mathbb{1}_{d=2} + n^{\alpha}\mathbb{1}_{d\geq 3}, & (3.1.57) \end{aligned}$$

where $\gamma_2 \in (0, 1/6), \gamma_3 > 9$. Then, $\mathbb{P}$-a.s., as $n \to \infty$,

$$S_n^{J,b} \Rightarrow V_\nu, \tag{3.1.58}$$

where V_ν is a subordinator with Lévy measure $\nu(u,\infty) = \mathcal{K}u^{-\alpha}$, where $\mathcal{K} = \mathcal{K}(d,\alpha,\theta) > 0$, and zero drift. Convergence holds weakly in the space $D[0,\infty)$ equipped with Skorohod's J_1 topology.

By analogy to Section 3.1.3, Theorem 3.7 allows to deduce the following results on the aging behavior of X.

Theorem 3.8. *Let $d \geq 2$. Under the assumptions of Theorem 3.7, $\mathbb{P}$-a.s., for $i = 1,2,3$*

$$\lim_{s\to\infty} \mathcal{C}_s^i(1,\rho) = \mathrm{Asl}_\alpha(1/(1+\rho)). \tag{3.1.59}$$

Similar dynamics of trap models were studied previously in [34]. There, under different conditions on J, it is proved that $\mathcal{C}_s^1(1,\rho)$ (and another correlation function) converges, in $\mathbb{P}$-law, respectively under some additional conditions in $\mathbb{P}$-probability to the distribution function of the arcsine law. The main differences between their and our assumptions on J are that in [34] the aging behavior of X is only studied for transient J's and that in our setting we assume that π_Y is symmetric. However, in view of (3.1.55), respectively (3.1.1), it is natural to assume that π_Y is symmetric. To prove statements in $\mathbb{P}$-law, a law of large numbers for the number of distinct points that J visits in n steps is required in [34]. For statements in $\mathbb{P}$-probability, another law of large numbers, namely for the distinct number of points that are visited by J and an independent copy of it, J', is necessary in [34].

The remainder of the paper is structured as follows. Section 3.2 contains the proof of Theorem 3.1 and Theorem 3.3. In Section 3.3 we collect preparatory results for the proof of Theorem 3.4. The latter is carried out in Section 3.4. The proof of Theorem 3.7 is contained in Section 3.5. Finally, Section 3.6 contains the proofs of Theorem 3.5, Theorem 3.6, and Theorem 3.8. Two lemmata are proven in the Appendix.

Acknowledgement. We thank Pierre Mathieu for pointing out that the proof of (4.6.10) in an earlier version was incomplete. A.S. thanks Sebastian Andres for many fruitful discussions.

3.2 Proof of Theorem 3.1 and Theorem 3.3

We now come to the proofs of the abstract theorems of Section 3.1. We first prove Theorem 3.1. We then show that the conditions of Theorem 3.3 imply those of Theorem 3.1, thereby proving Theorem 3.3. Finally, we state a lemma which shows that the conditions of both theorems simplify when the mapping that maps $u > 0$ to $\nu(u, \infty)$ is continuous.

Proof of Theorem 3.1. As mentioned earlier, the proof is based on a result by Durrett and Resnick [33] that gives conditions for partial sum processes of dependent random variables to converge. We use this result in a specialized form suitable for our application that we take from [37], namely Theorem 2.1 p. 7.

Throughout we fix a realization $\omega \in \Omega$ of the random environment but do not make this explicit in the notation. We set

$$\widehat{S}_n^{J,b}(t) \equiv S_n^{J,b}(t) - S_n^{J,b}(0), \quad t > 0. \tag{3.2.1}$$

Condition (A-0) ensures that $\widehat{S}_n^{J,b} - S_n^{J,b}$ converges to zero, uniformly. Thus, we must show that under Conditions (A-1)-(A-4)

$$\widehat{S}_n^{J,b} \Rightarrow V_\nu. \tag{3.2.2}$$

This will follow if we can verify Conditions (D1)-(D3) of Theorem 2.1 in [37] for $\widehat{S}_n^{J,b}$.

For this, let $\{\mathcal{F}_{n,k}, k \geq 0\}$ be an array of sigma algebras, where for $k \geq 0$, $\mathcal{F}_{n,k}$ is generated by $\{\ell_s(x),\ s \leq \theta_n k,\ x \in \mathbb{Z}^d\}$. When J is continuous $\mathcal{F}_{n,k}$ is generated by $\{J(s),\ s \leq \theta_n k\}$, whereas when J is discrete $\mathcal{F}_{n,k}$ is generated by $\{J(i), e_i,\ i \leq \theta_n k\}$. Note that for $n, k \geq 1$, $Z_{n,k}^J$ is $\mathcal{F}_{n,k}$ measurable and $\mathcal{F}_{n,k-1} \subset \mathcal{F}_{n,k}$.

We first establish that Condition (D1) is satisfied. For $u > 0$ and $t > 0$ we define

$$\nu_n^{J,t}(u, \infty) \equiv \textstyle\sum_{k=1}^{k_n(t)-1} \mathcal{P}_\mu \left(Z_{n,k+1}^J > u \mid \mathcal{F}_{n,k} \right). \tag{3.2.3}$$

This conditions then states that for all $u > 0$ such that $\nu(\{u\}) = 0$ and $\nu(u, \infty) < \infty$ and all $t > 0$ we have in $\mathcal{P}_\mu$-probability

$$\lim_{n \to \infty} \nu_n^{J,t}(u, \infty) = t\nu(u, \infty). \tag{3.2.4}$$

By the Markov property, $\nu_n^{J,t}(u, \infty)$ can be rewritten as

$$\nu_n^{J,t}(u, \infty) = \textstyle\sum_{k=1}^{k_n(t)-1} \sum_{x \in \mathcal{V}} \mathbb{1}_{J(k\theta_n)=x} Q_n^u(x) = k_n(t) \sum_{x \in \mathcal{V}} \pi_n^{J,t}(x) Q_n^u(x), \tag{3.2.5}$$

where, for $x \in \mathcal{V}$,

$$\pi_n^{J,t}(x) \equiv (k_n(t))^{-1} \textstyle\sum_{k=1}^{k_n(t)-1} \mathbb{1}_{J(k\theta_n)=x}, \tag{3.2.6}$$

denotes the empirical measure induced by the sequence $\{J(k\theta_n), k = 1, \ldots, k_n(t) - 1\}$. Taking the expectation with respect to $\mathcal{P}_\mu$, (3.2.5) yields

$$\mathcal{E}_\mu \nu_n^{J,t}(u, \infty) = k_n(t) \textstyle\sum_{x \in \mathcal{V}} \mathcal{E}_\mu \left(\pi_n^{J,t}(x) \right) Q_n^u(x) = \nu_n^t(u, \infty). \tag{3.2.7}$$

Since (A-2) ensures that $\lim_{n \to \infty} \nu_n^t(u, \infty) = t\nu(u, \infty)$ it suffices to prove that

$$\lim_{n \to \infty} \mathcal{P}_\mu \left(\left| \nu_n^{J,t}(u, \infty) - \nu_n^t(u, \infty) \right| > \varepsilon \right) = 0, \qquad \forall \varepsilon > 0, \tag{3.2.8}$$

i.e. that we may replace $\pi_n^{J,t}$ by its mean value. We do this by means of a second order Chebyshev inequality. For $x, y \in \mathcal{V}$ and $k, j \in \mathbb{N}$ write

$$\bar{q}_{k,j}(x, y) \equiv P_\mu(J(k) = x, J(j) = y) \quad \text{and} \quad q_k(x, y) \equiv P_x(J(k) = y), \tag{3.2.9}$$

with the convention that $q_k(y) \equiv P_\mu(J(k)=y)$. Then, on the one hand,

$$\begin{aligned}\mathcal{E}_\mu\left(\nu_n^{J,t}(u,\infty)\right)^2 &= \textstyle\sum_{x\in\mathcal{V}}\left(Q_n^u(x)\right)^2\left[k_n(t)\pi_n^t(x)+2\sum_{k=1}^{k_n(t)-2}\sum_{j=k+1}^{k_n(t)-1}\bar{q}_{k\theta_n,j\theta_n}(x,x)\right]\\ &+ \textstyle 2\sum_{\substack{x,x'\in\mathcal{V}\\ x\neq x'}}Q_n^u(x)Q_n^u(x')\sum_{k=1}^{k_n(t)-2}\sum_{j=k+1}^{k_n(t)-1}\bar{q}_{k\theta_n,j\theta_n}(x,x')\,,\end{aligned} \tag{3.2.10}$$

and on the other hand,

$$\begin{aligned}\left(\mathcal{E}_\mu\nu_n^{J,t}(u,\infty)\right)^2 &\geq \textstyle 2\sum_{x\in\mathcal{V}}\left(Q_n^u(x)\right)^2\sum_{k=1}^{k_n(t)-2}\sum_{j=k+1}^{k_n(t)-1}q_{k\theta_n}(x)q_{j\theta_n}(x)\\ &+ \textstyle 2\sum_{\substack{x,x'\in\mathcal{V}\\ x\neq x'}}Q_n^u(x)Q_n^u(x')\sum_{k=1}^{k_n(t)-2}\sum_{j=k+1}^{k_n(t)-1}q_{k\theta_n}(x)q_{j\theta_n}(x').\end{aligned} \tag{3.2.11}$$

Combining (3.2.10) and (3.2.11), we obtain that

$$\begin{aligned}&\mathcal{E}_\mu\left(\nu_n^{J,t}(u,\infty)\right)^2-\left(\mathcal{E}_\mu\nu_n^{J,t}(u,\infty)\right)^2\\ \leq\;& \sigma_n^t(u,\infty)\\ +\;& \textstyle\sum_{x\in\mathcal{V}}\left(Q_n^u(x)\right)^2\sum_{k=1}^{k_n(t)-2}\sum_{j=k+1}^{k_n(t)-1}\left[\bar{q}_{k\theta_n,j\theta_n}(x,x)-q_{k\theta_n}(x)q_{j\theta_n}(x)\right]\\ +\;& \textstyle\sum_{\substack{x,x'\in\mathcal{V}\\ x\neq x'}}Q_n^u(x)Q_n^u(x')\sum_{k=1}^{k_n(t)-2}\sum_{j=k+1}^{k_n(t)-1}\left[\bar{q}_{k\theta_n,j\theta_n}(x,x')-q_{k\theta_n}(x)q_{j\theta_n}(x')\right]\\ \equiv\;& (I)+(II)+(III).\end{aligned} \tag{3.2.12}$$

By (A-3), (I) tends to zero as $n\to\infty$. To bound (II), we drop the terms involving $q_{k\theta_n}(x)q_{j\theta_n}(x)$, and use the Markov property to write

$$\begin{aligned}(II) &\leq \textstyle\sum_{x\in\mathcal{V}}\left(Q_n^u(x)\right)^2\sum_{k=1}^{k_n(t)-2}\sum_{j=k+1}^{k_n(t)-1}q_{k\theta_n}(x)P_x\left(J((j-k)\theta_n)=x\right)\\ &\leq \textstyle\sum_{x\in\mathcal{V}}\left(Q_n^u(x)\right)^2\sum_{k=1}^{k_n(t)-2}q_{k\theta_n}(x)\sum_{j=1}^{k_n(t)-1}P_x\left(J(j\theta_n)=x\right)\\ &\leq \textstyle k_n(t)\sum_{x\in\mathcal{V}}\left(Q_n^u(x)\right)^2\pi_n^t(x)\sum_{j=1}^{k_n(t)-1}P_x\left(J(j\theta_n)=x\right)\\ &\leq \textstyle\sigma_n^t(u,\infty)\sup_{x\in\mathcal{V}}\sum_{j=1}^{k_n(t)-1}P_x\left(J(j\theta_n)=x\right).\end{aligned} \tag{3.2.13}$$

By (A-1) and (A-3), $(II)\to 0$ as $n\to\infty$.

Let us now show, using (A-1) and (A-2), that also (III) vanishes. Fix $x\in\mathcal{V}$, $k\geq 1$, and $j\geq k+1$. For every $x'\neq x$ we bound the term $Q_n^u(x')$ by 1. Now

$$\textstyle\sum_{x':\,x'\neq x}\bar{q}_{k\theta_n,j\theta_n}(x,x')=P_\mu(J(k\theta_n)=x,J(j\theta_n)\neq x)\leq P_\mu(J(k\theta_n)=x), \tag{3.2.14}$$

and

$$\textstyle\sum_{x':\,x'\neq x}q_{k\theta_n}(x)q_{j\theta_n}(x')=P_\mu(J(k\theta_n)=x)P_\mu(J(j\theta_n)\neq x), \tag{3.2.15}$$

so that, combining (3.2.14) and (3.2.15),

$$\begin{aligned}(III) &\leq \textstyle\sum_{x\in\mathcal{V}}Q_n^u(x)\sum_{k=1}^{k_n(t)-2}P_\mu(J(k\theta_n)=x)\sum_{j=1}^{k_n(t)-1}P_\mu(J(j\theta_n)=x)\\ &\leq \textstyle\nu_n^t(u,\infty)\sup_{x\in\mathcal{V}}\sum_{j=1}^{k_n(t)-1}P_\mu(J(j\theta_n)=x).\end{aligned} \tag{3.2.16}$$

By (A-2), $\nu_n^t(u,\infty)$ converges as $n\to\infty$ to $t\nu(u,\infty)$, which is a finite number. Thus invoking (A-1), $(II)\to 0$ as $n\to\infty$. Inserting our bounds in (3.2.12), the variance of $\nu_n^{J,t}(u,\infty)$ tends to zero as $n\to\infty$. The verification of Condition (D1) is complete.

Next we show that Condition (D2) of Theorem 2.1 in [37] is satisfied. For $u>0$, $t>0$ we define

$$\sigma_n^{J,t}(u,\infty)\equiv\textstyle\sum_{k=1}^{k_n(t)-1}\sum_{x\in\mathcal{V}}\left(\mathcal{P}_\mu\left(Z_{n,k+1}^J>u|\mathcal{F}_{n,k}\right)\right)^2. \tag{3.2.17}$$

This condition then states that for all $u > 0$ such that $\nu(\{u\}) = 0$ and $\nu(u, \infty) < \infty$, and all $t > 0$,

$$\lim_{n\to\infty} \mathcal{P}_\mu \left(\sigma_n^{J,t}(u,\infty) > \varepsilon\right) = 0, \qquad \forall\varepsilon > 0. \tag{3.2.18}$$

By the Markov property,

$$\sigma_n^{J,t}(u,\infty) = k_n(t) \sum_{x\in\mathcal{V}} \pi_n^{J,t}(x) \left(Q_n^u(x)\right)^2 . \tag{3.2.19}$$

The expectation of $\sigma_n^{J,t}(u,\infty)$ with respect to $\mathcal{P}_\mu$ is equal to $\sigma_n^t(u,\infty)$ and tends by (A-3) tends to zero. Thus, Condition (D2) is satisfied.

It remains to verify Condition (D3) of Theorem 2.1 in [37]. It is in particular satisfied if

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \sum_{k=1}^{k_n(t)-1} \mathcal{E}_\mu Z_{n,k+1}^J \mathbb{1}_{Z_{n,k+1}^J \leq \varepsilon} = 0. \tag{3.2.20}$$

By the Markov property the left hand side of (3.2.20) is equal to the left hand side of (3.1.22) and vanishes by (A-4). This proving that Condition (D3) is satisfied. Therefore, the conditions of Theorem 2.1 in [37] are verified, and so $\widehat{S}_n^{J,b} \Rightarrow V_\nu$ where convergence holds weakly in the space $D[0,\infty)$ equipped with Skorohod's J_1 topology and V_ν is a subordinator with Lévy measure ν and zero drift. □

In the verification of Condition (D1) of Theorem 2.1 in [37], more precisely in the proof of the claim (II), $(III) \to 0$, one sees that Condition (A-1) is used to replace $\pi_n^{J,t}$ by its average over P_μ. This is to be contrasted with the setting of [28] where (II) and (III) vanish because J is already in the invariant measure after θ_n steps, and hence for $x, x' \in \mathcal{V}$ and $j > k$ the event $\{J(k\theta_n) = x\}$ is essentially independent of $\{J(j\theta_n) = x'\}$.

Proof of Theorem 3.3. As in the proof of Theorem 3.1 we show that for given sequences a_n, c_n, θ_n, a given initial distribution μ and for fixed $\omega \in \Omega$ (B-2)-(B-5) $\Rightarrow$ (A-2)-(A-4). Since both Theorems require the conditions to be satisfied $\mathbb{P}$-a.s. for all $t > 0$ and all $u > 0$ such that $\nu(\{u\}) = 0$ and $\nu(u,\infty) < \infty$, it suffices to consider a fixed realization $\omega \in \Omega$ and fixed $u > 0$, $t > 0$. Let us first establish that, under the assumptions of Theorem 3.3,

$$\lim_{n\to\infty} \left|\nu_n^t(u,\infty) - \overline{\nu}_n^t(u,\infty)\right| = 0. \tag{3.2.21}$$

By (B-2), (3.2.21) implies (A-2). Next

$$\begin{aligned} & \left|\nu_n^t(u,\infty) - \overline{\nu}_n^t(u,\infty)\right| \\ \leq\ & \sum_{(x,k)\in\mathcal{A}_n^1} |P(J(k\theta_n) = x) - h_{k\theta_n}(x)| Q_n^u(x) \\ +\ & \sum_{(x,k)\in\mathcal{A}_n^2} |P(J(k\theta_n) = x) - h_{k\theta_n}(x)| Q_n^u(x). \end{aligned} \tag{3.2.22}$$

By (3.1.25) of (B-5) the second summand tends to zero. The first summand is smaller than

$$\begin{aligned} & \sup_{(x,k)\in\mathcal{A}_n^2} \frac{|P(J(k\theta_n) = x) - h_{k\theta_n}(x)|}{h_{k\theta_n}(x)} \sum_{(x,k)\in\mathcal{A}_n^2} h_{k\theta_n}(x) Q_n^u(x) \\ \leq\ & \sup_{(x,k)\in\mathcal{A}_n^2} \frac{|P(J(k\theta_n) = x) - h_{k\theta_n}(x)|}{h_{k\theta_n}(x)} \nu_n^t(u,\infty), \end{aligned} \tag{3.2.23}$$

and (3.1.24) of (B-5) guarantees that it vanishes as $n \to \infty$, proving that (A-2) is satisfied. To establish that

$$\lim_{n\to\infty} \left|\sigma_n^t(u,\infty) - \overline{\sigma}_n^t(u,\infty)\right| = 0, \tag{3.2.24}$$

we proceed as in (3.2.22). Bounding $Q_n^u(x) \leq 1$, the claim of (3.2.24) follows from (3.2.22)-(3.2.23) and (A-3) is satisfied as well. Condition (A-4) follows in a similar way. This finishes the proof of Theorem 3.3. □

Proof of Lemma 3.2. Let us show that (3.1.18) and (3.1.19) are always satisfied for transient x and never for positive recurrent x. Let $x \in \mathcal{V}$ be transient. Then we know that for any increasing sequence $\theta_n \gg 1$,

$$\lim_{n\to\infty} \int_{\theta_n}^{\infty} P_\mu(J(t) = x)dt = \lim_{n\to\infty} \int_{\theta_n}^{\infty} P_x(J(t) = x)dt = 0. \tag{3.2.25}$$

When J is a discrete time Markov chain this already proves (3.1.18) and (3.1.19). Let J be a continuous time Markov chain. Then we have for all initial distributions μ' and all $s < t$,

$$\begin{aligned} P_{\mu'}(J(t) = x) &\geq P_{\mu'}(J(s) = x)P_x(J(u) = x,\ \forall 0 < u \leq t - s) \\ &= P_{\mu'}(J(s) = x)\exp(-(t-s)\widetilde{\lambda}^{-1}(x)). \end{aligned} \tag{3.2.26}$$

Let $\mu' \in \{\delta_x, \mu\}$. We derive from (3.2.26) that

$$\begin{aligned} \sum_{k=1}^{k_n(t)-1} P_{\mu'}(J(k\theta_n) = x) &= \sum_{k=1}^{k_n(t)-1} \int_{k\theta_n}^{k\theta_n+1} P_{\mu'}(J(k\theta_n) = x)dt \\ &\leq \sum_{k=1}^{k_n(t)-1} \int_{k\theta_n}^{k\theta_n+1} e^{\widetilde{\lambda}^{-1}(x)} P_{\mu'}(J(t) = x)dt \\ &\leq e^{\widetilde{\lambda}^{-1}(x)} \int_{\theta_n}^{\infty} P_{\mu'}(J(t) = x)dt, \end{aligned} \tag{3.2.27}$$

which by (3.2.25) tends to zero. This proves that (3.1.18) and (3.1.19) hold for transient $x \in \mathcal{V}$.

Let $x \in \mathcal{V}$ be positive recurrent. When J is a continuous time chain, Theorem 1.8.3 in [55] states that $\lim_{t\to\infty} P_x(J(t) = x) = \pi(x) > 0$, where π is the invariant probability measure of J. When J is a discrete time chain with period $q \geq 1$, $\lim_{n\to\infty} P_x(J(nq) = x) = q/m_x > 0$ by Theorem 1.8.5 in [55], where m_x is the mean return time to x. Since $k_n(t) \gg 1$, (3.1.19) can only be satisfied if $P_x(J(t) = x)$, respectively $P_x(J(nd) = x)$, tend to zero, which contradicts that their limit is strictly positive.

Finally, since the existence of an invariant probability measure is equivalent to all $x \in \mathcal{V}$ being positive recurrent (see Theorem 3.5.3 in [55] for continuous time J that are non-explosive, respectively Theorem 1.7.7 in [55] for discrete time J), (A-1) cannot be satisfied when there exists an invariant probability measure. □

We show now that, when the measure ν is such that the mapping that maps u to $\nu(u,\infty)$ is continuous, the verification of the conditions of Theorems 3.1 and 3.3 simplifies.

Lemma 3.9. *Let ν be a sigma finite measure on $(0,\infty)$ such that $u \mapsto \nu(u,\infty)$ is continuous. Suppose that for given a_n, c_n, θ_n, μ and fixed $u > 0$, $t > 0$ there exists $\Omega^\tau(u,t) \subseteq \Omega$ with $\mathbb{P}(\Omega^\tau(u,t)) = 1$ such that, on $\Omega^\tau(u,t)$, (A-0)-(A-4), respectively (B-2)-(B-5), are verified. Then, for these sequences and this initial distribution (A-0)-(A-4), respectively (B-2)-(B-5), are satisfied $\mathbb{P}$-a.s. for all $u > 0$, $t > 0$.*

Proof. Since the proofs are the same, we only prove the claim for (A-0)-(A-4). Assume that (A-0)-(A-4) are satisfied $\mathbb{P}$-a.s. for fixed $u > 0$, $t > 0$ and given a_n, c_n, θ_n, and μ. We construct a set $\Omega^\tau \subseteq \Omega$ of full measure on which (A-0)-(A-4) are satisfied for all $u > 0$, $t > 0$. The sums on the right hand sides of (3.1.18), (3.1.19), (3.1.22), and the quantities $\nu_n^t(u,\infty)$, and $\sigma_n^t(u,\infty)$ depend on t through $k_n(t)\pi_n^t(x)$, $x \in \mathcal{V}$, which is increasing in t. Moreover, as sums of tail distributions, the quantities $\mathcal{P}_\mu(S_n^{J,b}(0) > u)$, $\nu_n^t(u,\infty)$, and $\sigma_n^t(u,\infty)$ are decreasing in u. Also, the right hand sides of (3.1.18)-(3.1.22) are continuous in t, respectively u. Thus, $\Omega^\tau \equiv \bigcap_{u,t>0,u,t\in\mathbb{Q}} \Omega^\tau(u,t) \subseteq \Omega$ is of full measure and (A-0)-(A-4) hold true for all $u > 0$, $t > 0$ on Ω^τ. The proof of Lemma 3.9 is finished. □

3.3 Application to BATM

This section and the next are devoted to the proof of Theorem 3.4. In the present section we derive new conditions that imply (B-2)-(B-5) and are specific to BATM. We also show that (A-0) and (A-1) hold true for BATM. In Section 3.4 we prove that these new conditions are satisfied and give the conclusion of the proof.

3.3.1 The VSRW

We collect results for $\widetilde{J}$ that are used in the proof of Theorem 3.4. The VSRW is a well-studied Markov jump process in random environment (see [4], [2], [31], [1], and the references therein). The proof of Theorem 3.4 relies heavily on very precise results for $\widetilde{J}$ that can be found in [4]. The results that we are using repeatedly concern the heat kernel, which we now define. For $x, y \in \mathbb{Z}^d$ and $t > 0$ the heat kernel is given by

$$q_t(x, y) \equiv P_x(\widetilde{J}(t) = y). \tag{3.3.1}$$

The bounds for $q_t(x, y)$ that are contained in [4] allow us to control all hitting, local, and exit times of vertices and balls that we need for the proof of Theorem 3.4. Moreover, we use the local central limit theorem which can be found in [4]. Note that in virtue of Theorem 6.1 in [4] and Lemma 9.1 in [2], these theorems apply in the present setting. We denote by $|\cdot|$ the Euclidian distance. For convenience, we restate Theorem 1.2 (a)-(c) (heat kernel bounds) and Theorem 5.14 (local central limit theorem) from [4].

Theorem 3.10. *There exists $c_1 \in (0, \infty)$ such that for all $x, y \in \mathbb{Z}^d$ and $t > 0$,*

$$q_t(x, y) \leq c_1 t^{-d/2}. \tag{3.3.2}$$

There exist identically distributed random variables $\{U_x\}_{x\in\mathbb{Z}^d}$ whose distribution satisfies

$$\mathbb{P}(U_x > v) \leq c_1 \exp(-c_2 v^{1/3}), \quad v > 0, \tag{3.3.3}$$

where $c_1, c_2 \in (0, \infty)$, and such that we have

$$q_t(x, y) \leq c_1 t^{-d/2} e^{-c_2|x-y|\{1\wedge|x-y|t^{-1}\}}, \quad \text{if } |x-y| \vee t^{1/2} \geq U_x, \tag{3.3.4}$$
$$q_t(x, y) \geq c_1 t^{-d/2} e^{-c_2|x-y|^2 t^{-1}}, \quad \text{if } t \geq U_x^2 \vee |x-y|^{4/3}. \tag{3.3.5}$$

For $x \in \mathbb{R}^d$ write $\lfloor x\rfloor = (\lfloor x_1\rfloor, \dots, \lfloor x_d\rfloor)$. There exists $c_v > 0$ such that, for $T > 0$,

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}^d} \sup_{t\geq T} \left| n^{d/2} q_{nt}(0, \lfloor n^{1/2}x\rfloor) - (2\pi c_v t)^{d/2} e^{-|x|^2/2c_v t}\right| = 0, \quad \mathbb{P}\text{-a.s.}. \tag{3.3.6}$$

By Lemma 3.3 in [2], there exists $c_0 \in (0, \infty)$ and $n_0 = n_0(\omega)$ with $\mathbb{P}(n_0 < \infty) = 1$ such that on $\{\omega : n \geq n_0\}$,

$$\sup_{x:\ |x|\leq a_n} U_x \leq c_0(\log a_n)^3. \tag{3.3.7}$$

Therefore, whenever we apply (3.3.4) and (3.3.5) of Theorem 3.10 we check whether, given $x, y \in \mathbb{Z}^d$ and $t > 0$, $|x - y| \wedge t^{1/2} \geq c_0(\log a_n)^3$.

We now state two lemmata that are needed in the proof of Theorem 3.4. Their proofs are postponed to the appendix. The first concerns the distribution of the exit times of certain balls. We denote by $B_r(x)$ the ball of radius r centered at x; by convention $B_r \equiv B_r(0)$. We write $\eta(B_r(x))$ for the exit time of $B_r(x)$.

Lemma 3.11. *Let a_n be as in* (3.1.39). *There exists $c_4 \in (0,\infty)$ such that the following holds. For all sequences m_n, r_n such that $m_n \geq c_0^2 r_n (\log a_n)^6$ and $a_n \geq m_n$, $\mathbb{P}$-a.s.,*

$$P_x(\eta(B_{r_n}(x)) \leq m_n) \leq e^{-c_4 r_n^2 m_n^{-1}}, \quad \forall x \in B_{a_n}. \tag{3.3.8}$$

For all sequences m_n, r_n such that $r_n \geq c_0 (\log a_n)^3$ and $m_n \geq 3r_n^2$, $\mathbb{P}$-a.s.,

$$P_x(\eta(B_{r_n}(x)) \geq m_n) \leq e^{-c_4 m_n^{1/2} r_n^{-1}}, \quad \forall x \in B_{a_n}. \tag{3.3.9}$$

The second lemma provides bounds on the expected number of different sites that $\widetilde{J}$ visits in certain time intervals. Given an increasing sequence of integers, m_n we define the range of $\widetilde{J}$ in the time interval $[0, m_n]$ as

$$R_{m_n} \equiv \textstyle\sum_y \mathbb{1}_{\sigma(y) \leq m_n}, \tag{3.3.10}$$

where $\sigma(y) \equiv \inf\{t \geq 0 : \widetilde{J}(t) = y\}$ is the hitting time of y.

Lemma 3.12. *Let m_n be such that $a_n \geq m_n \geq c_0^2 (\log a_n)^6$. There exists $c_5 \in (0,\infty)$ such that for $d \geq 3$, we have for all $x \in B_{a_n}$,*

$$\mathbb{E}E_x\left(R_{m_n}\right)^k \leq c_5\left(m_n \mathbb{1}_{k=1} + \sqrt{m_n}\mathbb{1}_{k=2}\right). \tag{3.3.11}$$

For $d = 2$ there exists $f_{m_n} : (0,\infty) \to (0,\infty)$ such that, $\mathbb{P}$-a.s., for all $x \in B_{a_n}$, $y \in \mathbb{Z}^d$,

$$\begin{aligned} &P_x(\sigma(y) \leq m_n) \leq f_{m_n}(|x-y|), \\ \textit{and} \qquad &E_x\left(R_{m_n}\right)^k \leq \textstyle\sum_y \left(f_{m_n}(|x-y|)\right)^k \leq c_5 m_n (\log m_n)^{-k}. \end{aligned} \tag{3.3.12}$$

Notice that by our choices of θ_n we may use Lemma 3.12 for $m_n \geq \theta_n^\delta$ for $\delta \geq 2/3$.

3.3.2 Specializing Theorem 3.3 for BATM

In this section we specialize Theorem 3.3 to the setting of BATM. In order to prove Theorem 3.4, i.e. to obtain $\mathbb{P}$-a.s. convergence on time scales $c_n = n$, we proceed as in [16] (see proof of Lemma 3.1, p. 2366) and consider sequences of the form $\exp((N+r)^k)$ for $N \in \mathbb{N}$, $r \in [0,1]$ and $k = 7\gamma_3/(1-\alpha)$ first. In the sequel we denote by $\overset{J_1}{\Longrightarrow}$, respectively $\overset{M_1}{\Longrightarrow}$, weak convergence in the space $D[0,\infty)$ equipped with Skorohod's J_1 topology, respectively Skorohod's M_1 topology.

Lemma 3.13. *Suppose that there exists $\Omega^\tau \subseteq \Omega$ with $\mathbb{P}(\Omega^\tau) = 1$ such that on Ω^τ, uniformly in r, $S^{\widetilde{J},b}_{\exp((N+r)^k)} \overset{J_1}{\Longrightarrow} V_\alpha$ as $N \to \infty$. Then on Ω^τ, $S_N^{\widetilde{J},b} \overset{J_1}{\Longrightarrow} V_\alpha$ as $N \to \infty$.*

Proof. Fix $\omega \in \Omega^\tau$. The above assumption can be rewritten as follows. For any $\varepsilon > 0$ there exists $N^\star \in \mathbb{N}$ and $\delta > 0$ such that

$$\mathcal{P}_0(\rho_\infty(S^{\widetilde{J},b}_{\exp((N+r)^k)}, V_\alpha) > \varepsilon) \leq \delta, \quad \forall r \in [0,1], \quad \forall N \geq N^\star, \tag{3.3.13}$$

where ρ_∞ is Skorohod's J_1 metric on $D[0,\infty)$. Taking for $N \in \mathbb{N}$

$$r = (\log N)^{1/k} - \lfloor (\log N)^{1/k} \rfloor \in [0,1], \tag{3.3.14}$$

we have $N = e^{\left(\lfloor (\log N)^{1/k}\rfloor + r\right)^k}$. Since $\lfloor (\log N)^{1/k} \rfloor \in \mathbb{N}$ we find by (3.3.13)-(3.3.14) that

$$\mathcal{P}_0(\rho_\infty(S_N^{\widetilde{J},b}, V_\alpha) > \varepsilon) \leq \delta, \quad \forall \lfloor (\log N)^{1/k} \rfloor \geq N^\star. \tag{3.3.15}$$

In other words $S_N^{\widetilde{J},b} \Rightarrow V_\alpha$ as $N \to \infty$ for all $\omega \in \Omega^\tau$. $\square$

From now on we assume that n is given by

$$n = \exp((N+r)^k), \quad \text{where } k = 7\gamma_3/(1-\alpha), \tag{3.3.16}$$

and take the limit $N \to \infty$. For $c_n = \exp((N+r)^k)$ and fixed $r \in [0,1]$ we construct a sequence of subsets $\Omega_N(r)$ with $\mathbb{P}((\Omega_N(r))^c) \le c(r)N^{-2}$ such that the following holds. On $\Omega_N(r)$, for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\mathcal{P}_0(\rho_\infty(S^{\widetilde{J},b}_{\exp((N+r)^k}, V_\alpha) > \varepsilon) \le \delta. \tag{3.3.17}$$

By Borel-Cantelli Lemma we get that, $\mathbb{P}$-a.s., for fixed $r \in [0,1]$, $S^{\widetilde{J},b}_{\exp((N+r)^k)} \overset{J_1}{\Longrightarrow} V_\alpha$ as $N \to \infty$. Since the bound on $\mathbb{P}((\Omega_N(r))^c)$ will be derived from Lemma 3.11 and Lemma 3.12, which are valid for all r and are independent of r, one can show that $c(r)$ is monotone in r. Hence, by the same arguments as in the proof of Lemma 3.9, the convergence of $S^{\widetilde{J},b}_{\exp((N+r)^k)}$ is uniform in $r \in [0,1]$ and we obtain by Lemma 3.13 that, $\mathbb{P}$-a.s., $S^{\widetilde{J},b}_N \overset{J_1}{\Longrightarrow} V_\alpha$ as $N \to \infty$. Thus, we may assume throughout the rest of the paper that $r = 0$.

We will not study $S^{\widetilde{J},b}_n$ directly, but another process, $\overline{S}^{\widetilde{J},b}_n$, to which only those x contribute for which $\tau(x)$ is 'large enough'. More precisely, for $x \in \mathbb{Z}^d$ we set

$$\gamma_n(x) \equiv c_n^{-1}\tau(x). \tag{3.3.18}$$

Let $\epsilon_n(d) \equiv (\log\theta_n)^{-2}\mathbb{1}_{d=2} + \theta_n^{-1/3}\mathbb{1}_{d\ge3}$ and denote by $T_n \equiv \{y \in \mathbb{Z}^d : \gamma_n(y) > \epsilon_n\}$ the collection of 'large' traps. Then,

$$\overline{S}^{\widetilde{J},b}_n(t) \equiv \textstyle\sum_{k=0}^{k_n(t)-1} \sum_{x\in\mathbb{Z}^d} \gamma_n(x)\mathbb{1}_{x\in T_n}(\widetilde{\ell}_{\theta_n(k+1)}(x) - \widetilde{\ell}_{\theta_n k}(x)), \quad t > 0, \tag{3.3.19}$$

where $\widetilde{\ell}_t(y) = \int_0^t \mathbb{1}_{\widetilde{J}(s)=y} ds$.

Roughly speaking, the following lemma states that, $\mathbb{P}$-a.s., $\overline{S}^{\widetilde{J},b}_n$ is a good approximation for $S^{\widetilde{J},b}_n$. To simplify notation, we write $P \equiv P_0$, respectively $\mathcal{P} \equiv \mathcal{P}_0$.

Lemma 3.14. $\mathbb{P}$*-a.s.,* $\lim_{\varepsilon\to0}\limsup_{N\to\infty} \mathcal{P}(\rho_\infty(S^{\widetilde{J},b}_n, \overline{S}^{\widetilde{J},b}_n) > \varepsilon) = 0.$

Proof. By definition of ρ_∞ it suffices to show this result with ρ_∞ replaced by ρ_r, Skorohod's J_1 metric on $D[0,r]$, for all $r > 0$. For convenience we take $r = 1$ and we get

$$\mathcal{P}\Big(\rho_\infty\Big(S^{\widetilde{J},b}_n, \overline{S}^{\widetilde{J},b}_n\Big) > \varepsilon\Big) = \mathcal{P}\Big(S^{\widetilde{J},b}_n(1) - \overline{S}^{\widetilde{J},b}_n(1) > \varepsilon\Big). \tag{3.3.20}$$

By Chebyshev's inequality the right hand side of (3.3.20) is bounded from above by

$$\varepsilon^{-1}\mathcal{E}\textstyle\int_0^{a_n} \gamma_n(\widetilde{J}(s))\mathbb{1}_{\widetilde{J}(s)\notin T_n} ds = \varepsilon^{-1}\sum_{y\notin T_n} \mathcal{E}\widetilde{\ell}_{a_n}(y)\gamma_n(y). \tag{3.3.21}$$

The lemma will be proven if we can show that the expectation of (3.3.21) with respect to the random environment,

$$\varepsilon^{-1}\textstyle\sum_{x\in\mathbb{Z}^d} \mathbb{E}\Big[\mathcal{E}\widetilde{\ell}_{a_n}(x)\gamma_n(x)\mathbb{1}_{\gamma_n(x)\le\epsilon_n}\Big], \tag{3.3.22}$$

tends to zero fast enough. We decompose the sum in (3.3.22) into three sums according to the size of $|x|$. Namely, we set $A_1 \equiv \{x \in \mathbb{Z}^d : |x| \le c_0(\log a_n)^3\}$, $A_2 \equiv \{x \in \mathbb{Z}^d : c_0(\log a_n)^3 < |x| \le a_n^{1/2}\log\log a_n\}$, and $A_3 \equiv \{x \in \mathbb{Z}^d : |x| > a_n^{1/2}\log\log a_n\}$. When $x \in A_1$, we know by (3.3.2) of Theorem 3.10 that $\mathcal{E}\widetilde{\ell}_{a_n}(x) \leq c_1 \log a_n$ $\mathbb{P}$-a.s.. Since moreover $\mathbb{E}\gamma_n(0)\mathbb{1}_{0\notin T_n} \leq c\, c_n^{-\alpha}\epsilon_n^{1-\alpha}$ for some $c \in (0,\infty)$, we have

$$\tfrac{1}{\varepsilon}\textstyle\sum_{x\in A_1}\mathbb{E}\left[\mathcal{E}\widetilde{\ell}_{a_n}(x)\gamma_n(x)\mathbb{1}_{x\notin T_n}\right] \leq \tfrac{c_1}{\varepsilon}\textstyle\sum_{x\in A_1}\log a_n \mathbb{E}\left[\gamma_n(x)\mathbb{1}_{x\notin T_n}\right] \leq c_1 c\, c_n^{-\alpha/2}. \quad (3.3.23)$$

For $x \in A_3$ we derive from (3.3.4) of Theorem 3.10 that $\mathcal{E}\widetilde{\ell}_{a_n}(x) \leq e^{-c_2|x|^2/a_n}$ $\mathbb{P}$-a.s., and get

$$\begin{aligned}\tfrac{1}{\varepsilon}\textstyle\sum_{x\in A_3}\mathbb{E}\left[\mathcal{E}\widetilde{\ell}_{a_n}(x)\gamma_n(x)\mathbb{1}_{x\notin T_n}\right] &\leq \tfrac{c}{\varepsilon}\textstyle\sum_{x\in A_3} e^{-c_2|x|^2/a_n}c_n^{-\alpha}\epsilon_n^{1-\alpha}\\ &= \tfrac{c}{\varepsilon}\textstyle\sum_{k>a_n^{1/2}\log\log a_n} k^{d-1}e^{-c_2k^2/a_n}c_n^{-\alpha}\epsilon_n^{1-\alpha}\\ &\leq e^{-c_2/2(\log\log a_n)^2}. \end{aligned}\quad (3.3.24)$$

Finally, let $x \in A_2$. By (3.3.4) of Theorem 3.10 we know that, $\mathbb{P}$- a.s., $\mathcal{E}\widetilde{\ell}_{a_n}(x) \leq c_3|x|^{2-d}$ if $d \geq 3$ and $\mathcal{E}\widetilde{\ell}_{a_n}(x) \leq c_2 \log a_n$ if $d = 2$. Thus we have

$$\begin{aligned}\tfrac{1}{\varepsilon}\textstyle\sum_{x\in A_2}\mathbb{E}\left[\mathcal{E}\widetilde{\ell}_{a_n}(x)\gamma_n(x)\mathbb{1}_{x\notin T_n}\right] &= \tfrac{c}{\varepsilon}\textstyle\sum_{x\in A_2} c_n^{-\alpha}\epsilon_n^{1-\alpha}\left(|x|^{2-d}\mathbb{1}_{d\geq3}+\log a_n\mathbb{1}_{d=2}\right)\\ &= \tfrac{c}{\varepsilon}c_n^{-\alpha}\epsilon_n^{1-\alpha}\textstyle\sum_{k=1}^{a_n^{1/2}\log\log a_n} k\left(\mathbb{1}_{d\geq3}+\log a_n\mathbb{1}_{d=2}\right)\\ &\leq \tfrac{c}{\varepsilon}\epsilon_n^{1-\alpha}(\log\log a_n)^2\left(\mathbb{1}_{d\geq3}+(\log\theta_n)^{1-\alpha}\mathbb{1}_{d=2}\right)\\ &= \tfrac{c'}{\varepsilon}\log N\left(N^{-12/d}\mathbb{1}_{d\geq3}+N^{-3}\mathbb{1}_{d=2}\right) \end{aligned}\quad (3.3.25)$$

where $c, c' \in (0,\infty)$ and where the last line follows from (3.1.38), (3.1.39), and (3.3.16). Collecting (3.3.23)-(3.3.25), the proof of Lemma 3.14 is complete. □

Using Theorem 3.3, we can derive new conditions for the process $\overline{S}_n^{\widetilde{J},b}$ to converge. To present these conditions, we introduce the following quantities. For $x, y \in \mathbb{Z}^d$, $u > 0$, and $\varepsilon > 0$ we define

$$Q_n^u(x,y) \equiv \mathcal{P}_x\left(\widetilde{\ell}_{\theta_n}(y)\gamma_n(y) > u, \eta(B_{\theta_n}(x)) > \theta_n\right)\mathbb{1}_{y\in T_n}, \quad (3.3.26)$$

$$M_n^\varepsilon(x,y) \equiv \mathcal{E}_x\left(\widetilde{\ell}_{\theta_n}(y)\gamma_n(y)\mathbb{1}_{\gamma_n(y)\widetilde{\ell}_{\theta_n}(y)\leq\varepsilon}\mathbb{1}_{\eta(B_{\theta_n}(x))>\theta_n}\right)\mathbb{1}_{y\in T_n}. \quad (3.3.27)$$

Note that $Q_n^u(0,y) = M_n^\varepsilon(x,y) = 0$ for $y \notin B_{\theta_n}(x)$. For $t > 0$ we set $d_n(t) \equiv \lfloor a_n t\rfloor^{1/2}\log\lfloor a_n t\rfloor$. For $n \in \mathbb{N}$ and $x \in \mathbb{Z}^d$ we take $h_{k\theta_n}(x) = \mathbb{E}P(J(n) = x)$. Thus, $\overline{\pi}_n^t(x) = \mathbb{E}\pi_n^t(x)$. By analogy to (3.1.28) and (3.1.29) we write, for $u > 0$, $t > 0$,

$$\widetilde{\nu}_n^t(u,\infty) \equiv k_n(t)\textstyle\sum_{x\in B_{d_n(t)}}\overline{\pi}_n^t(x)\sum_{y\in\mathbb{Z}^d}Q_n^u(x,y), \quad (3.3.28)$$

and

$$\widetilde{\sigma}_n^t(u,\infty) \equiv k_n(t)\textstyle\sum_{x\in B_{d_n(t)}}\overline{\pi}_n^t(x)\sum_{y\in\mathbb{Z}^d}(Q_n^u(x,y))^2. \quad (3.3.29)$$

We also define for $\varepsilon > 0$, $t > 0$

$$m_n^t(\varepsilon) \equiv k_n(t)\textstyle\sum_{x\in B_{d_n(t)}}\overline{\pi}_n^t(x)\sum_{y\in\mathbb{Z}^d}M_n^\varepsilon(x,y), \quad (3.3.30)$$

and finally we introduce for $\varepsilon > 0$ the set

$$\mathcal{B}_n \equiv \{(x,k) \in B_{d_n(t)} \times [k_n(t)-1]\setminus\{0\} : |x|^2 < \varepsilon k\theta_n,\ |x|^2 > k\theta_n/\varepsilon\}. \quad (3.3.31)$$

We are now ready to present our new conditions. They are stated for fixed $\omega \in \Omega$.

(C-2) For all $u > 0$, $t > 0$

$$\lim_{N\to\infty}\widetilde{\nu}_n^t(u,\infty) = t\mathcal{K}u^{-\alpha}. \quad (3.3.32)$$

(C-3) For all $u > 0, t > 0$

$$\lim_{N\to\infty} \widetilde{\sigma}_n^t(u,\infty) = 0. \tag{3.3.33}$$

(C-4) For all $t > 0$ there exists $C(t) > 0$ such that for all $\varepsilon > 0$

$$\limsup_{N\to\infty} m_n^t(\varepsilon) \le C(t)\varepsilon^{1-\alpha}. \tag{3.3.34}$$

(C-5) For all $u > 0$, $t > 0$, $\varepsilon > 0$ there exists $C(u,t) \in (0,\infty)$ and $N(\varepsilon)$ such that for $n \ge N(\varepsilon)$,

$$\sum_{(x,k)\in\mathcal{B}_n} \sum_y (k\theta_n)^{-d/2} e^{-c_2|x|^2/k\theta_n} Q_n^u(x,y) \le C(u,t)\varepsilon, \tag{3.3.35}$$

$$\sum_{(x,k)\in\mathcal{B}_n} \sum_{|y-x|\le\theta_n} (k\theta_n)^{-d/2} e^{-c_2|x|^2/k\theta_n} M_n^\varepsilon(x,y) \le C(u,t)\varepsilon. \tag{3.3.36}$$

Proposition 3.15. *Assume that Conditions (C-2)-(C-5) are satisfied $\mathbb{P}$-a.s. for fixed $u > 0$, $t > 0$, and $\varepsilon > 0$. Then, $\mathbb{P}$-a.s., $\overline{S}_n^{\widetilde{J},b} \overset{J_1}{\Longrightarrow} V_\alpha$, as $N \to \infty$.*

The next lemma is designed to control quantities that appear in the course of the proof of Proposition 3.15 (namely, in the treatment of $\widetilde{\nu}_n^t(u,\infty)$, $\widetilde{\sigma}_n^t(u,\infty)$, and $m_n^t(\varepsilon)$) when considering the following two (kind of) events. The first is that, given $x \in B_{d_n(t)}$, one can find y that belongs to $B_{\theta_n}(x) \cap T_n$ and $y' \in B_{\theta_n}(x)$ such that $\gamma_n(y') > c_n^{-1} m_n$, where $m_n > \theta_n^a$ for suitable $a > 0$.

Lemma 3.16. *For all $t > 0$ and m_n such that $m_n \ge \theta_n^a$ for $a > 2d/\alpha$ there exists $K(t) \in (0,\infty)$ such that, for N large enough,*

$$k_n(t) \sum_{x\in B_{d_n(t)}} \overline{\pi}_n^t(x) \mathbb{E}\left[\sum_{y,y':|x-y|,|x-y'|\le\theta_n} \mathbb{1}_{y\in T_n} \mathbb{1}_{\gamma_n(y')>c_n^{-1}m_n}\right] \le K(t)\frac{\theta_n^{2d}}{m_n^\alpha}. \tag{3.3.37}$$

We first prove Proposition 3.15 assuming Lemma 3.16 and the lemma next.

Proof of Proposition 3.15. Let us apply Theorem 3.3 to $\overline{S}_n^{\widetilde{J},b}$. By Lemma 3.9 it suffices to prove that the conditions of Theorem 3.3 are verified $\mathbb{P}$-a.s. for fixed $u > 0$, $t > 0$, and $\varepsilon > 0$.

We first prove that Conditions (A-0) and (A-1) are satisfied for $\overline{S}_n^{\widetilde{J},b}$. To verify Condition (A-0), fix $u > 0$. Since $S_n^{\widetilde{J},b}(0) = 0$, unless there is $y \in T_n$ for which $\widetilde{\ell}_{\theta_n}(0) > 0$,

$$\mathcal{P}(\overline{S}_n^{\widetilde{J},b}(0) > u) \le \mathcal{P}(\eta(B_{\theta_n^{3/4}}) \le \theta_n) + \sum_{y:|y|\le\theta_n^{3/4}} \mathbb{1}_{y\in T_n}. \tag{3.3.38}$$

By (3.3.8) of Lemma 3.11, the first term on the right hand side of (3.3.38) tends $\mathbb{P}$-a.s. to zero. The second term on the right hand side of (3.3.38) is, when taking expectation with respect to the random environment, bounded above by $\theta_n^{3d/4} c_n^{-\alpha} \epsilon_n^{-\alpha}$. By our choice of c_n and θ_n this is summable in N and hence Condition (A-0) is satisfied. To verify (A-1), we use (3.3.2) of Theorem 3.10 which yields for all $x, y \in \mathbb{Z}^d$ and all $t > 0$,

$$\sum_{k=1}^{k_n(t)-1} q_{k\theta_n}(x,y) \le \sum_{k=1}^{k_n(t)-1} c_1(\theta_n k)^{-d/2} \le \theta_n^{-1} \sum_{k=1}^{k_n(t)-1} c_1 k^{-1} \le 2c_1 \frac{\log a_n t}{\theta_n}. \tag{3.3.39}$$

By (3.1.38), (3.1.39), and (3.3.16) this is smaller than N^{-2}, and hence (A-1) is satisfied.

We now establish that (C-2)-(C-5) $\Rightarrow$ (B-2)-(B-5). First we prove that $|\overline{\nu}_n^t(u,\infty) - \widetilde{\nu}_n^t(u,\infty)|$ tends $\mathbb{P}$-a.s. to zero, i.e. that (C-2) $\Rightarrow$ (B-2). Observe that

$$\begin{aligned} |\overline{\nu}_n^t(u,\infty) - \widetilde{\nu}_n^t(u,\infty)| &\le k_n(t) \sum_{x\notin B_{d_n(t)}} \overline{\pi}_n^t(x) \\ &+ k_n(t) \sum_{x\in B_{d_n(t)}} \overline{\pi}_n^t(x) P_x(\eta(B_{\theta_n}(x)) \le \theta_n) \\ &+ k_n(t) \sum_{x\in B_{d_n(t)}} \overline{\pi}_n^t(x) \sum_{y,y':|x-y|,|x-y'|\le\theta_n} \mathbb{1}_{y,y'\in T_n} \\ &= (I) + (II) + (III). \end{aligned} \tag{3.3.40}$$

Let us now prove that (I)-(III) tend $\mathbb{P}$-a.s. to zero. By the definition of $\overline{\pi}_n^t(x)$, and by (3.3.8) of Lemma 3.11 we have, $\mathbb{P}$-a.s.,

$$\begin{aligned}(I) \quad &= \textstyle\sum_{k=1}^{k_n(t)-1} \sum_{x\notin B_{d_n(t)}} \mathbb{E}P(\widetilde{J}(k\theta_n)=x) \leq \sum_{k=1}^{k_n(t)-1} \mathbb{E}P(\eta(B_{d_n(t)}) \leq k\theta_n) \\ &\leq \textstyle\sum_{k=1}^{k_n(t)-1} \mathbb{E}P(\eta(B_{(k\theta_n)^{3/4}}) \leq k\theta_n) \quad \leq k_n(t)e^{-c_4\sqrt{\theta_n}}, \end{aligned} \tag{3.3.41}$$

proving that, $\mathbb{P}$-a.s., $(I) \to 0$. Also by (3.3.8) of Lemma 3.11, $\mathbb{P}$-a.s.,

$$P_x(\eta(B_{\theta_n}(x)) \leq \theta_n) \leq P_x(\eta(B_{\theta_n^{3/4}}(x)) \leq \theta_n) \leq \exp(-c_4\sqrt{\theta_n}), \tag{3.3.42}$$

and hence, $\mathbb{P}$-a.s., $(II) \leq k_n(t)e^{-c_4\sqrt{\theta_n}}$, which tends to zero. Finally, by a first order Chebyshev inequality,

$$\mathbb{P}\left((III) > \varepsilon\right) \leq \tfrac{k_n(t)}{\varepsilon} \textstyle\sum_{x\in B_{d_n(t)}} \overline{\pi}_n^t(x) \sum_{y,y':|x-y|,|x-y'|\leq\theta_n} \mathbb{E}(\mathbb{1}_{y,y'\in T_n}) \leq \frac{K(t)\theta_n^{2d}}{(c_n\epsilon_n)^{\alpha}}, \tag{3.3.43}$$

where we used Lemma 3.16. This is summable in N, and so, $\mathbb{P}$-a.s., $(III) \to 0$. Therefore, (C-2) $\Rightarrow$ (B-2). In a similar way one can show (C-3) $\Rightarrow$ (B-3).

We now prove (C-4) $\Rightarrow$ (B-4).Observe that

$$\begin{aligned} &\textstyle\sum_{x\in\mathbb{Z}^d} \overline{\pi}_n^t(x)\mathcal{E}_x\Big[\overline{Z}_{n,1}^{\widetilde{J}}\mathbb{1}_{\overline{Z}_{n,1}^{\widetilde{J}}\leq\varepsilon}\Big] \\ &\leq (I)+(II)+\textstyle\sum_{x\in B_{d_n(t)}} \overline{\pi}_n^t(x)\mathcal{E}_x\Big[\overline{Z}_{n,1}^{\widetilde{J}}\mathbb{1}_{\overline{Z}_{n,1}^{\widetilde{J}}\leq\varepsilon}\mathbb{1}_{\eta(B_{\theta_n}(x))>\theta_n}\Big]. \end{aligned} \tag{3.3.44}$$

By (3.3.41) and (3.3.42), $(I), (II) \to 0$. Denoting by (III') the last term in the right hand side of (3.3.44), it remains to establish that $(III') \leq m_n^t(\varepsilon)$. Since

$$(III') = \textstyle\sum_{x\in B_{d_n(t)}} \overline{\pi}_n^t(x) \sum_{y\in\mathbb{Z}^d\cap T_n} \mathcal{E}_x\Big[\gamma_n(y)\widetilde{\ell}_{\theta_n}(y)\mathbb{1}_{\overline{Z}_{n,1}^{\widetilde{J}}\leq\varepsilon}\mathbb{1}_{\eta(B_{\theta_n}(x))>\theta_n}\Big], \tag{3.3.45}$$

and since $\{\overline{Z}_{n,1}^{\widetilde{J}} \leq \varepsilon\} \subset \{\widetilde{\ell}_{\theta_n}(y)\gamma_n(y) \leq \varepsilon\}$ for all $y \in \mathbb{Z}^d$ for which $\widetilde{\ell}_{\theta_n}(y) > 0$, we have $(III') \leq m_n^t(\varepsilon)$. Thus, (C-4) $\Rightarrow$ (B-4).

Finally we prove that (C-5) $\Rightarrow$ (B-5). The local central limit theorem, (3.3.6) of Theorem 3.10, implies that, $\mathbb{P}$-a.s., for $\mathcal{A}_n^1 = \{(x,k) : k \geq 1, |x|^2 \in (\varepsilon k\theta_n, k\theta_n/\varepsilon)\}$

$$\lim_{N\to\infty} \sup_{(x,k)\in\mathcal{A}_n^1} |(k\theta_n)^{d/2}q_{k\theta_n}(x) - (2\pi c_v)^{d/2}e^{-|x|^2/(2c_v k\theta_n T)}| = 0, \tag{3.3.46}$$

where $q_t(x) \equiv q_t(0,x)$. By (3.3.3) of Theorem 3.10, $(k\theta_n)^{d/2}q_{k\theta_n}(x) \leq c_1$ for all $x \in \mathbb{Z}^d$, $k \in \mathbb{N}$ and so by the bounded convergence theorem,

$$\lim_{N\to\infty} \sup_{(x,k)\in\mathcal{A}_n^1} |(k\theta_n)^{d/2}h_{k\theta_n}(x) - (2\pi c_v)^{d/2}e^{-|x|^2/(2c_v k\theta_n T)}| = 0, \tag{3.3.47}$$

where $h_n(x) = \mathbb{E}q_n(x)$. Thus, there exists $N = N(\varepsilon)$ such that, for $n \geq N(\varepsilon)$, $\mathbb{P}$-a.s.,

$$\sup_{(x,k)\in\mathcal{A}_n^1} \frac{|q_{k\theta_n}(x) - h_{k\theta_n}(x)|}{h_{k\theta_n}(x)} \leq \sqrt{\varepsilon}, \tag{3.3.48}$$

showing that $\mathbb{P}$-a.s. (3.1.24) is satisfied for $\mathcal{A}_n^1$. Hence, it suffices to verify (3.1.25) and (3.1.26) for the set $\mathcal{A}_n^2 \equiv \mathbb{Z}^d \times [k_n(t)-1] \setminus \mathcal{A}_n^1$. The set $\mathcal{A}_n^2$ is the disjoint union of $[k_n(t)-1] \times \mathbb{Z}^d \setminus B_{d_n(t)}$ and $\mathcal{B}_n$. Let us now verify (3.1.25) and (3.1.26) for each of the sets separately. Since, $\mathbb{P}$-a.s., $(I) \to 0$, we know that $[k_n(t)-1] \times \mathbb{Z}^d \setminus B_{d_n(t)}$ satisfies (3.1.25) and (3.1.26). By (3.3.4) of Theorem 3.10 we have, $\mathbb{P}$-a.s., $q_{k\theta_n}(x) \leq c_1(k/\theta_n)^{-d/2}e^{-c_2|x|^2/k\theta_n}$ for $x \in B_{d_n(t)}$ and $k \geq 1$ and therefore (C-5) implies (3.1.25) and (3.1.26) for $\mathcal{B}_n$.Thus, (C-5) $\Rightarrow$ (B-5). The proof of Proposition 3.15 is complete. $\square$

Proof of Lemma 3.16. Since the τ's are identically distributed and since $\overline{\pi}_n^t$ is a probability measure it suffices to prove that

$$k_n(t)\mathbb{E}\left[\sum_{y,y':|y|,|y'|\le\theta_n} \mathbb{1}_{y\in T_n}\mathbb{1}_{\gamma_n(y')>c_n^{-1}m_n}\right] \le K(t)m_n^{-\alpha}\theta_n^{2d}, \tag{3.3.49}$$

Now observe that

$$k_n(t)\sum_{y,y':|y|,|y'|\le\theta_n} \mathbb{E}\left(\mathbb{1}_{y\in T_n}\mathbb{1}_{\gamma_n(y')>c_n^{-1}m_n^a}\right) \le k_n(t)(c_n\epsilon_n)^{-\alpha}\theta_n^{2d}m_n^{-\alpha}, \tag{3.3.50}$$

which for our choice of a_n, c_n, and θ_n (cf. (3.1.38) and (3.1.39)) is smaller than $K(t)m_n^{-\alpha}\theta_n^{2d}$. $\square$

3.4 Verification of Conditions (C-2)-(C-5)

In this section we show that (C-2)-(C-5) are satisfied. Let $u > 0$, $t > 0$ and $\varepsilon > 0$ be fixed. In Section 3.4.1 we establish that $\lim_{N\to\infty}\mathbb{E}\widetilde{\nu}_n^t(u,\infty) = t\nu(u,\infty)$. Next, in Section 3.4.2, we bound the variance of $\widetilde{\nu}_n^t(u,\infty)$ by a quantity that is summable in N. In Section 3.4.3 we prove that $\lim_{N\to\infty}\mathbb{E}\widetilde{\sigma}_n^t(u,\infty) = 0$ and show that the convergence speed is summable. In Section 3.4.4 we establish that $\mathbb{E}m_n^t(\varepsilon) \le C(t)\varepsilon^{1-\alpha}$ and show that the variance of $m_n^t(\varepsilon)$ is summable in N. We verify Condition (C-5) in Section 3.4.4. Finally, we conclude the proof of Theorem 3.3 in Section 3.4.5.

3.4.1 Convergence of $\mathbb{E}\widetilde{\nu}_n^t(u,\infty)$

This section is devoted to the proof of convergence of $\mathbb{E}\widetilde{\nu}_n^t(u,\infty)$, which is the most demanding part of the proof of Theorem 3.3.

Lemma 3.17. *For all $u > 0$, $t > 0$, $\lim_{N\to\infty}\mathbb{E}\widetilde{\nu}_n^t(u,\infty) = t\nu(u,\infty)$.*

Proof of Lemma 3.17. Since the τ's are identically distributed we have

$$\sum_{y\in\mathbb{Z}^d}\mathbb{E}Q_n^u(x,y) = \sum_{y\in\mathbb{Z}^d}\mathbb{E}Q_n^u(0,y), \quad \forall x\in B_{d_n}(t). \tag{3.4.1}$$

The statement of the lemma is thus equivalent to

$$\lim_{N\to\infty} k_n(t)\sum_{y\in\mathbb{Z}^d}\mathbb{E}Q_n^u(0,y) = t\nu(u,\infty), \quad \forall u>0,\ \forall t>0. \tag{3.4.2}$$

In view of (3.3.26), the sum in (3.4.2) is over $y\in B_{\theta_n}$. In fact, we can restrict it to $y\in B_{\theta_n}\setminus\{0\}$ because $\mathbb{E}Q_n(0,0) \le \mathbb{E}(\mathbb{1}_{0\in T_n}) = c_n^{-\alpha}\epsilon_n^{-\alpha} \ll k_n(t)$. To prove (3.4.2) we distinguish two cases depending on whether $d\ge 3$ or $d=2$.

Case 1. Let $d\ge 3$ and take $y\in B_{\theta_n}\setminus\{0\}$. Set $k(\theta_n) = \theta_n(\log\theta_n)^{-1}$ and $h(\theta_n) = \theta_n - k(\theta_n)$. By the Markov property, writing $f_{\sigma(y)}$ for the density function of the hitting time, $\sigma(y)$, of y

$$\begin{aligned}
&\mathcal{P}(\widetilde{\ell}_{\theta_n}(y)\gamma_n(y) > u, \eta(B_{\theta_n}) > \theta_n)\\
\ge\ & \int_0^{\theta_n} f_{\sigma(y)}(t)\mathcal{P}_y(\widetilde{\ell}_{\theta_n - t}(y)\gamma_n(y) > u)dt - P(\eta(B_{\theta_n}) \le \theta_n)\\
\ge\ & \int_0^{h(\theta_n)} f_{\sigma(y)}(t)dt\, \mathcal{P}_y(\widetilde{\ell}_{k(\theta_n)}(y)\gamma_n(y) > u) - e^{-c_4\theta_n^{1/2}}\\
=\ & P(\sigma(y)\le h(\theta_n))\,\mathcal{P}_y(\widetilde{\ell}_{k(\theta_n)}(y)\gamma_n(y) > u) - e^{-c_4\theta_n^{1/2}},
\end{aligned} \tag{3.4.3}$$

where we used (3.3.8) of Lemma 3.11 in the second step. We first deal with the second probability in (3.4.3). Setting $B_n^1 \equiv B_{\sqrt{k(\theta_n)}(\log\theta_n)^{-2}}(y)$ we have,

$$\begin{aligned}
\mathcal{P}_y(\widetilde{\ell}_{k(\theta_n)}(y)\gamma_n(y) > u) \ &\ge\ \mathcal{P}_y(\widetilde{\ell}_{\eta(B_n^1)}(y)\gamma_n(y) > u, \eta(B_n^1) < k(\theta_n))\\
&\ge\ \mathcal{P}_y(\widetilde{\ell}_{\eta(B_n^1)}(y)\gamma_n(y) > u) - \mathcal{P}_y(\eta(B_n^1)\ge k(\theta_n)).
\end{aligned} \tag{3.4.4}$$

By (3.3.9) of Lemma 3.11 the second term in (3.4.4) is smaller than $e^{-c_4(\log\theta_n)^2}$. To bound the first term in (3.4.4) we use the fact that when $\widetilde{J}$ starts in y, $\widetilde{\ell}_{\eta(B_n^1)}(y)$ is exponentially distributed. This well-known fact can be seen as follows. Let $N_n^1(y)$ be the number of successive visits of $\widetilde{J}$ to y in the time interval $[0,\eta(B_n^1)]$, when $\widetilde{J}(0)=y$. Then

$$\widetilde{\ell}_{\eta(B_n^1)}(y)=\sum_{i=1}^{N_n^1(y)} e_i\widetilde{\lambda}^{-1}(y), \tag{3.4.5}$$

where $\{e_i, i\in\mathbb{N}\}$ is a collection of i.i.d. mean one exponential random variables, independent of $N_n^1(y)$. Note that the sum in (3.4.5) starts in one because $N_n^1(y)\geq 1$. Since $\eta(B_n^1)$ is a stopping time one can show, using the strong Markov property that $N_n^1(y)$ has a geometric distribution. Therefore, $\widetilde{\ell}_{\eta(B_n^1)}(y)$ is a sum of a geometric number of i.i.d. exponentials, which in turn is exponentially distributed. Let

$$g_{B_n^1}(y)\equiv E_y\left[\int_0^{\eta(B_n^1(y))}\mathbb{1}_{\widetilde{J}(s)=y}ds\right] \tag{3.4.6}$$

denote the mean value of $\widetilde{\ell}_{\eta(B_n^1}(y)$. Eq. (3.4.3) then becomes

$$\mathcal{P}_y(\widetilde{\ell}_{k(\theta_n)}(y)\gamma_n(y)>u)\geq e^{-u\left(\gamma_n(y)g_{B_n^1}(y)\right)^{-1}}-e^{-c_4(\log\theta_n)^2}, \tag{3.4.7}$$

and we get

$$Q_n^u(0,y)\geq P(\sigma(y)\leq h(\theta_n))\big(e^{-u(\gamma_n(y)g_{B_n^1}(y))^{-1}}-e^{-c_4(\log\theta_n)^2}\big)\mathbb{1}_{y\in T_n}. \tag{3.4.8}$$

To get an upper bound we write (using the Markov property)

$$\begin{aligned}\mathcal{P}(\widetilde{\ell}_{\theta_n}(y)\gamma_n(y)>u,\eta(B_{\theta_n})>\theta_n) &\leq \mathcal{P}(\widetilde{\ell}_{\theta_n}(y)\gamma_n(y)>u)\\ &= \int_0^{\theta_n} f_{\sigma(y)}(t)\mathcal{P}_y(\widetilde{\ell}_{\theta_n-t}(y)\gamma_n(y)>u)dt\\ &\leq P\left(\sigma(y)\leq\theta_n\right)\mathcal{P}_y(\widetilde{\ell}_{\theta_n}(y)\gamma_n(y)>u). \end{aligned}\tag{3.4.9}$$

Set $B_n^2\equiv B_{\sqrt{\theta_n}\log\theta_n}(y)$. By (3.3.8) of Lemma 3.11 we know that $\widetilde{J}$ exits B_n^2 before time θ_n with a probability smaller than $e^{-c_4(\log\theta_n)^2}$. Thus, proceeding as in (3.4.4)

$$Q_n^u(0,y)\leq P\left(\sigma(y)\leq\theta_n\right)\big(e^{-u(\gamma_n(y)g_{B_n^2}(y))^{-1}}+e^{-c_4(\log\theta_n)^2}\big)\mathbb{1}_{y\in T_n} \tag{3.4.10}$$

The contribution to $\mathbb{E}\widetilde{\nu}_n^t(u,\infty)$ coming from the error terms $\exp(-c_4(\log\theta_n)^2)$ in (3.4.8) and (3.4.10) is negligible because

$$k_n(t)\sum_{|y|\leq\theta_n}\mathbb{E}\big[\mathbb{1}_{y\in T_n}e^{-c_4(\log\theta_n)^2}\big]\ll e^{-c_4/2(\log\theta_n)^2}. \tag{3.4.11}$$

To calculate $\mathbb{E}Q_n^u(0,y)$, we distinguish whether $\theta>0$ or $\theta=0$. In the first case several objects depend on the random environment: the distribution of $\sigma(y)$, the mean local time $g_{B_n^i}(y)$, and $\gamma_n(y)$. Thus we first seek upper and lower bounds on the distribution of $\sigma(y)$ and on $g_{B_n^i}(y)$ that are independent of $\gamma_n(y)$. Moreover, we look for upper and lower bounds for $g_{B_n^i}(y)$ that are independent on N.

Let us begin with bounds for $P\left(\sigma(y)\leq\theta_n\right)$. We show now that we may approximate the distribution of $\sigma(y)$ by that of $\min_{y'\sim y}\sigma(y')$, which is independent of $\gamma_n(y)$. Since $y\neq 0$ we know that $\min_{y'\sim y}\sigma(y')\leq\sigma(y)$, implying that

$$P(\sigma(y)\leq\theta_n)\leq P(\min_{y'\sim y}\sigma(y')\leq\theta_n). \tag{3.4.12}$$

Define the event $D(y) = \{\exists\, y' :\ |y - y'| \le 2, \tau(y) > \theta_n^a\}$, where $a \in (0,\infty)$. By Lemma 3.16 we can choose a such that

$$k_n(t) \sum_{|y|\le\theta_n} \mathbb{E}\big[P(\sigma(y) \le h(\theta_n))\mathbb{1}_{y\in T_n}\mathbb{1}_{D(y)}\big] \le K(t)\theta_n^{-\varepsilon}, \tag{3.4.13}$$

for some $\varepsilon > 0$. Consequently, we may assume that all the traps in the neighborhood $y \in T_n$ have size smaller than θ_n^a. This implies that, as soon as $\widetilde{J}$ visits a neighbor y' of y, it jumps to y with probability larger than $1 - 2d(\theta_n^a c_n^{-1})^\theta$. This term goes to 1 when $\theta > 0$ and we get, for all $\varepsilon > 0$ and $y' \sim y$, that

$$P_{y'}(\varepsilon < \sigma(y) \le h(\theta_n)) \le \theta_n^{a\theta} c_n^{-\theta} \ll c_n^{-\theta/2}. \tag{3.4.14}$$

Thus,

$$P\left(\sigma(y) \le h(\theta_n)\right)\mathbb{1}_{y\in T_n}\mathbb{1}_{D(y)} \ge \left(P\left(\min_{y'\sim y}\sigma(y') \le h(\theta_n)\right) - c_n^{-\theta/2}\right)\mathbb{1}_{y\in T_n}. \tag{3.4.15}$$

As in (3.4.11), we see that the contribution of the error $c_n^{-\theta/2}$ to $\mathbb{E}\widetilde{\nu}_n^t(u,\infty)$ is of order $o(1)$.

Let us now approximate $g_{B_n^i}(y)$ by random variables, $\widetilde{g}_\infty(y)$, that are independent of $\gamma_n(y)$. This approximation follows closely the ideas of [2]. For $i = 1,2$ we use the classical variational representation (see e.g. Chapter 3 in [27]) to write

$$(g_{B_n^i}(y))^{-1} = \inf\left\{\tfrac{1}{2}\sum_{x\sim z}\widetilde{\lambda}(x,z)(f(x)-f(z))^2 : f|_y = 1, f|_{B_n^i} = 0\right\}, \tag{3.4.16}$$

and define, setting $A(y) \equiv \{y' : y' \sim y\} \cup \{y\}$,

$$(\widetilde{g}_{B_n^i}(y))^{-1} \equiv \inf\left\{\tfrac{1}{2}\sum_{x\sim z}\widetilde{\lambda}(x,z)(f(x)-f(z))^2 : f|_{A(y)} = 1, f|_{B_n^i} = 0\right\}. \tag{3.4.17}$$

Let us show that for all $\varepsilon > 0$ there exists $N(\varepsilon)$ uniform in the realization of the random environment, such that for $N \ge N(\varepsilon)$, for all $y \in T_n \cap B_{\theta_n}$, on the event $(D(y))^c$,

$$\widetilde{g}_{B_n^i}(y) \le g_{B_n^i}(y) \le (1+\varepsilon)\widetilde{g}_{B_n^i}(y). \tag{3.4.18}$$

The proof of (3.4.18) is similar to that of Lemma 6.2 in [2] and we only sketch it here. The first inequality in (3.4.18) follows since the infimum in (3.4.16) is taken over a larger set of functions than in (3.4.17). For the second, we use the fact that the infimum in (3.4.16) is attained for the function $f(x) = P_x(\sigma(y) < \eta(B_n^i))$. Set $\varepsilon_n \equiv 2d(\theta_n^a c_n^{-1})^\theta$. We now construct a function $\overline{f}$ that satisfies $f \ge (1-\varepsilon_n)\overline{f}$ and that is admissible in (3.4.17). For this let $y^\star$ be such that $f(y^\star) = \min_{y'\sim y} f(y')$ and take $\overline{f}(\cdot) = 1 \wedge (f(\cdot)/f(y^\star))$. On $(D(y))^c \cap \{y \in T_n\}$ we have $f \ge (1-\varepsilon_n)\overline{f}$. To see this, recall that $p(y',y) \ge 1-\varepsilon_n$ for all $y' \sim y$. Hence, $f(y^\star) \ge 1-\varepsilon_n$ and consequently $f \ge (1-\varepsilon_n)\overline{f}$ on $(D(y))^c \cap \{y \in T_n\}$. This finishes the proof of (3.4.18). Combining (3.4.15), (3.4.18), and (3.4.8) we get that $\mathbb{E}\widetilde{\nu}_n^t(u,\infty)$ is bounded below by

$$k_n(t)\sum_{|y|\le\theta_n}\mathbb{E}\big[P(\min_{y'\sim y}\sigma(y') \le h(\theta_n))e^{-u(\gamma_n(y)\widetilde{g}_{B_n^1}(y))^{-1}}\mathbb{1}_{y\in T_n}\big] - o(1), \tag{3.4.19}$$

where we used once again (3.4.13) to bound the contributions to $\mathbb{E}\widetilde{\nu}_n^t(u,\infty)$ of y's in $B_{\theta_n} \cap T_n$ for which $D(y)$ occurs. Similarly, we obtain by (3.4.15), (3.4.13), (3.4.18), and (3.4.10) that $\mathbb{E}\widetilde{\nu}_n^t(u,\infty)$ is smaller than

$$k_n(t)\sum_{|y|\le\theta_n}\mathbb{E}\big[P(\min_{y'\sim y}\sigma(y') \le \theta_n)e^{-u(1-\varepsilon_n)(\gamma_n(y)\widetilde{g}_{B_n^2}(y))^{-1}}\mathbb{1}_{y\in T_n}\big] + o(1). \tag{3.4.20}$$

Let $g_\infty(y) = \lim_{N\to\infty} g_{B_n^1}(y) = \lim_{N\to\infty} g_{B_n^2}(y)$. By Lemma 3.5 in [2] we know that for all $\varepsilon' > 0$, $\mathbb{P}$-a.s., there exists $N(\varepsilon')$, uniform in the random environment, such that

$$(1-\varepsilon')g_\infty(y) \le g_{B_n^i}(y) \le g_\infty(y), \qquad \forall y \in B_{d_n(t)}, \qquad \forall N \ge N(\varepsilon'). \tag{3.4.21}$$

This with (3.4.18) implies that for all $\varepsilon'' > 0$ there exists $N(\varepsilon'')$ such that for $N \geq N(\varepsilon'')$, for all $y \in T_n \cap B_{\theta_n}$, on the event $(D(y))^c$, $(1-\varepsilon'')\widetilde{g}_\infty(y) \leq g_{B_n^i}(y) \leq (1+\varepsilon'')\widetilde{g}_\infty(y)$, where $\widetilde{g}_\infty(y) = \lim_{N\to\infty} \widetilde{g}_{B_n^1}(y) = \lim_{N\to\infty} \widetilde{g}_{B_n^2}(y)$. Equipped with (3.4.19) we take expectation with respect to $\gamma_n(y)$ and obtain

$$\mathbb{E}\widetilde{\nu}_n^t(u,\infty) \geq \tfrac{t(1-\varepsilon'')^\alpha \Gamma(1+\alpha,\epsilon_n)}{u^\alpha \theta_n} \textstyle\sum_{|y|\leq\theta_n} \mathbb{E}[\widetilde{g}_\infty^\alpha(y) P(\min_{y'\sim y}\sigma(y') \leq h(\theta_n))] - o(1). \quad (3.4.22)$$

As in the proof of Lemma 3.12 one sees that adding $|y| > \theta_n$ in (3.4.22) produces at most an error of the order of $e^{-c_4/2\theta_n}$, and so

$$\mathbb{E}\widetilde{\nu}_n^t(u,\infty) \geq \frac{t(1-\varepsilon'')^\alpha \Gamma(1+\alpha,\epsilon_n)}{u^\alpha \theta_n} \sum_{y\in\mathbb{Z}^d} \mathbb{E}[\widetilde{g}_\infty^\alpha(y) P(\min_{y'\sim y}\sigma(y') \leq h(\theta_n))] - o(1). \quad (3.4.23)$$

Similarly,

$$\mathbb{E}\widetilde{\nu}_n^t(u,\infty) \leq \frac{t\Gamma(1+\alpha)}{u^\alpha \theta_n} \sum_{y\in\mathbb{Z}^d} \mathbb{E}[\widetilde{g}_\infty^\alpha(y) P(\min_{y'\sim y}\sigma(y') \leq \theta_n)\,] + o(1). \quad (3.4.24)$$

Since $\epsilon_n \to 0$, $\Gamma(1+\alpha,\epsilon_n) \to \Gamma(1+\alpha)$. It remains to establish that

$$\lim_{N\to\infty} \theta_n^{-1} \textstyle\sum_{y\in\mathbb{Z}^d} \mathbb{E}[\widetilde{g}_\infty^\alpha(y) P(\min_{y'\sim y}\sigma(y') \leq h_i(\theta_n))] = K, \quad \text{for } i=1,2, \quad (3.4.25)$$

where $h_1(\theta_n) = \theta_n$ and $h_2(\theta_n) = h(\theta_n)$. Since $h_2 = h_1 - o(h_1)$, we only present the proof for h_1. For $\beta \in [0,1]$, set $f_n^\beta(x) \equiv \sum_{y\in\mathbb{Z}^d} \mathbb{E}\big[(\widetilde{g}_\infty(y)/c_6)^\beta P(\min_{y'\sim y}\sigma(y') \leq n)\big]$, where $c_6 \in (0,\infty)$ is such that $c_6 \geq g_\infty(y) \geq \widetilde{g}_\infty(y)$ for all $y \in \mathbb{Z}^d$. Using a 'quasi' sub-additivity argument (see (3.4.27) below), we now establish that $\lim_{n\to\infty} f_n^\beta/n = K'$ where $K' = \inf\{f_n^\beta/n : n \in \mathbb{N}\} \in (0,\infty)$. First note that by Lemma 3.12 $f_n^\beta/n \leq f_n^0/n \leq 2dc_5$, and so, $K' < \infty$. To see that $K' > 0$, we use that $f_m^\beta \geq f_m^1$ and bound f_m^1 from below:

$$\textstyle\sum_{y\in\mathbb{Z}^d} g_\infty(y) P(\sigma(y) \leq m) \geq \sum_{y\in\mathbb{Z}^d} E\widetilde{\ell}_m(y) \geq m. \quad (3.4.26)$$

As in the proof of (3.4.18) one can show that $\widetilde{g}_\infty(y) \geq (2d)^2 g_\infty(y)$, and hence $f_m^1 \geq (2d)^{-2}m$, which proves that $K' > 0$. Let us now assume that for all $\varepsilon > 0$ there exists N large enough such that for all $n, m \geq N$,

$$f_{n+m}^\beta \leq (1+\varepsilon) f_m^\beta + f_n^\beta. \quad (3.4.27)$$

Then convergence to K' follows. Indeed, by construction of K' there exists M such that $f_M^\beta/M < K' + \varepsilon/2$. Now, let $N^\star = N'M$, $N' \geq N$, be such that $f_{2M}^\beta/N^\star < \varepsilon/2$. For $n \geq N^\star$ write $n = sM + r \geq N^\star$ where $s \geq N$, $r \leq M$. Then, by (3.4.27),

$$f_n^\beta/n \leq (1+\varepsilon)\tfrac{s-1}{n} f_M^\beta + f_{M+r}^\beta/n \leq (1+\varepsilon)\tfrac{s-1}{s+2} f_M^\beta/M + f_{2M}^\beta/N^\star \leq (1+2\varepsilon)K' - \varepsilon. \quad (3.4.28)$$

Thus, f_n^β/n converges to K' because by construction, $f_n^\beta/n \geq K'$. It remains to establish the claim of (3.4.27). The difference $f_{n+m}^\beta - f_n^\beta$ is equal to

$$\begin{aligned} &\textstyle\sum_{y\in\mathbb{Z}^d} \mathbb{E}\big[(\tfrac{\widetilde{g}_\infty(y)}{c_6})^\beta P(\min_{y'\sim y}\sigma(y') \in (n, n+m))\big] \\ &= \textstyle\sum_{z,y} \mathbb{E}\big[(q_n(z) - \mathbb{E}q_n(z))(\tfrac{\widetilde{g}_\infty(y)}{c_6})^\beta P_z(\min_{y'\sim y}\sigma(y') \leq m)\big] + f_m^\beta. \end{aligned} \quad (3.4.29)$$

The first summand on the right hand side of (3.4.29) is smaller than εf_m^β if

$$\varepsilon_m^\beta \equiv \textstyle\sum_z \mathbb{E}\big[|q_n(z) - \mathbb{E}q_n(z)| \sum_y P_z(\sigma(y) \leq m)\big] \leq \varepsilon m. \quad (3.4.30)$$

We divide the sum into $z \in B_{n^{1/2}/\varepsilon'}$ and $z \notin B_{n^{1/2}/\varepsilon'}$. Let $z \in B_{n^{1/2}/\varepsilon'}$. From the proof of Lemma 3.12 we know that there exists $c'' \in (0,\infty)$ such that

$$\sum_{y\in\mathbb{Z}^d} P_z(\sigma(y) \le m) \le c'' \sum_{|z-y|\le m^{1/2}\log m} |y|^{2-d} e^{-c_4/2|y|^2/m} (g_\infty(y))^{-1}. \tag{3.4.31}$$

We bound $1/g_\infty(y) \le 1/\varepsilon' + (g_\infty(y))^{-1}\mathbb{1}_{g_\infty(y)<\varepsilon'}$ and call (I), respectively (II) the contribution to ε_m^β coming from $1/\varepsilon'$, respectively $(g_\infty(y))^{-1}\mathbb{1}_{g_\infty(y)<\varepsilon'}$, and $z \in B_{n^{1/2}/\varepsilon'}$. Now,

$$(I) = c''m1/\varepsilon' \sum_{|z|\le n^{1/2}/\varepsilon'} \mathbb{E}\big[|q_n(z) - \mathbb{E}q_n(z)|/\mathbb{E}q_n(z)\big]\mathbb{E}q_n(z). \tag{3.4.32}$$

Since $q_n(z) \le c_1 n^{-d/2}$, the contribution to (I) from $z \in B_{\varepsilon' n^{1/2}}$ is smaller than $(\varepsilon')^{d-1}m$. By (3.3.6) of Theorem 3.10 for $z \in B_{n^{1/2}/\varepsilon'} \setminus B_{\varepsilon' n^{1/2}}$, $\mathbb{E}\big[|q_n(z) - \mathbb{E}q_n(z)|/\mathbb{E}q_n(z)\big]$ tends uniformly to zero, and so (I) is bounded above by εm. Also,

$$(II) \le c_1 m(\varepsilon')^{-d} c\mathbb{E}\big[(g_\infty(0))^{-1}\mathbb{1}_{g_\infty(0)<1/\varepsilon'}\big] \le c_1 m(\varepsilon')^{-d}\exp(-c/\varepsilon'^{1/3}) < \varepsilon m, \tag{3.4.33}$$

where we used that $(g_\infty(0))^{-1} \le U_0^{1/(d-2)}$ and (3.3.3). Let $z \notin B_{n^{1/2}/\varepsilon'}$. The summands in ε_m^β depend on z only through $|z|$. Thus, writing z_k for z such that $z \in \partial B_k$, the contribution to ε_m^β coming from $z \notin B_{n^{1/2}/\varepsilon'}$ is smaller than

$$\sum_{k>n^{1/2}/\varepsilon'} \mathbb{E}\big[[P(\widetilde{J}(n) \in \partial B_k) + \mathbb{E}P(\widetilde{J}(n) \in \partial B_k)] \sum_y P_{z_k}(\sigma(y) \le m)\big]. \tag{3.4.34}$$

By (3.3.8) of Lemma 3.11, $P(\widetilde{J}(n) \in \partial B_k) \le e^{-c_4 k/n^2}$, which shows together with Lemma 3.12 that (3.4.34) is smaller than $e^{-c_4/\varepsilon'}m$. This proves (3.4.27). Thus (3.4.25) holds, and so,

$$u^{-\alpha}tK\Gamma(1+\alpha)(1+\varepsilon)^\alpha + o(1) \ge \mathbb{E}\widetilde{\nu}_n^t(u,\infty) \ge u^{-\alpha}tK\Gamma(1+\alpha,\epsilon_n)(1-\varepsilon)^\alpha - o(1). \tag{3.4.35}$$

Since $\varepsilon > 0$ is arbitrary we see that $\lim_{N\to\infty} \mathbb{E}\widetilde{\nu}_n^t(u,\infty) = u^{-\alpha}t\mathcal{K}$, where $\mathcal{K} \equiv K\Gamma(1+\alpha)$. This concludes the proof of Lemma 3.17 for $d \ge 3$ and $\theta > 0$. When $\theta = 0$, the proof simplifies because $\widetilde{J}$ is independent of the random environment. More precisely, it suffices to use Lemma 3.5 in [2] to replace $g_{B_n^i}(y)$ by $g_\infty(y)$ to get

$$\begin{aligned}\tfrac{t\Gamma(1+\alpha,\epsilon_n)}{u^\alpha\theta_n} &\sum_{y\in\mathbb{Z}^d} g_\infty(y)^\alpha P\left(\sigma(y) \le h(\theta_n)\right) - o(1)\\ &\le \mathbb{E}\widetilde{\nu}_n^t(u,\infty) \le \tfrac{t\Gamma(1+\alpha)(1+\varepsilon')^\alpha}{u^\alpha\theta_n} \sum_{y\in\mathbb{Z}^d} g_\infty^\alpha(y)P(\sigma(y) \le \theta_n) + o(1). \end{aligned}\tag{3.4.36}$$

By the same arguments as for $\theta > 0$, we can show that both bounds converge to $u^{-\alpha}t\mathcal{K}$ as $N \to \infty$. This finishes the proof of Lemma 3.17 for $d \ge 3$.

Case 2. Let $d = 2$. The pattern of proof is similar to that of Case 1 and relies on (3.4.3) and (3.4.9). The difference lies in the behavior $g_{B_n^i}(y)$. By definition in (3.4.6),

$$g_{B_n^i}(y) = \int_0^\infty P_y(J(t) = y, \eta(B_n^1) > t)dt \ge \int_{\sqrt{\theta_n}}^{\theta_n} P_y(J(t) = y)dt - e^{-c_4(\log\theta_n)^2}, \tag{3.4.37}$$

where we used (3.3.8) of Lemma 3.11. By (3.3.5) of Theorem 3.1, the integral on the right hand side of (3.4.37) is larger than $c_1/2\log\theta_n$, showing that $g_{B_n^i}(y)$ diverges as $N \to \infty$. Thus, instead of substituting $\widetilde{g}_\infty(y)$ for $g_{B_n^i}(y)$ we use (3.4.18) to approximate $g_{B_n^i}(y)$ by $\widetilde{g}_{B_n^i}(y)$ for N large enough. A number of results from [31] will allow us to deal with $\widetilde{g}_{B_n^i}(y)$.

Let us begin with the construction of a lower bound on $\mathbb{E}\widetilde{\nu}_n^t(u,\infty)$. We deduce from (3.4.15), that bounding $P(\sigma(y) \le h(\theta_n)) \ge P(\min_{y\sim y'}\sigma(y') \le h(\theta_n))$ for $y \in T_n$, produces in $\mathbb{E}\widetilde{\nu}_n^t(u,\infty)$ an error of the order $\theta_n^{-\varepsilon}$ for $\varepsilon > 0$. We use (3.4.18) to substitute $\widetilde{g}_{B_n^1}(y)$ for $g_{B_n^1}(y)$. Since

$P(\min_{y'\sim y}\sigma(y')\leq h(\theta_n))$ and $\widetilde{g}_{B_n^1}(y)$ are independent of $\gamma_n(y)$, we can proceed as in Case 1 and take expectation with respect to $\gamma_n(y)$. Doing this yields

$$\begin{aligned}\mathbb{E}\widetilde{\nu}_n^t(u,\infty) &\geq \frac{t\log\theta_n\Gamma(1+\alpha,\epsilon_n)}{(u\log\theta_n)^\alpha\theta_n}\sum_{|y|\leq\theta_n}\mathbb{E}[\widetilde{g}^\alpha_{B_n^1}(y)P(\min_{y'\sim y}\sigma(y')\leq h(\theta_n))]-o(1)\\ &\geq \frac{t\log\theta_n\Gamma(1+\alpha,\epsilon_n)}{(u\log\theta_n)^\alpha\theta_n}\sum_{|y|\leq\theta_n}\mathbb{E}[\widetilde{g}^\alpha_{B_n^1}(y)P(\sigma(y)\leq h(\theta_n))]-o(1), \qquad (3.4.38)\end{aligned}$$

since $\min_{y'\sim y}\sigma(y)\leq\sigma(y)$. We now construct an upper bound on $\mathbb{E}\widetilde{\nu}_n^t(u,\infty)$. Again, by (3.4.18) and since $\min_{y'\sim y}\sigma(y)\leq\sigma(y)$,

$$\mathbb{E}\widetilde{\nu}_n^t(u,\infty)\leq\frac{t\log\theta_n\Gamma(1+\alpha)(1+\varepsilon)^\alpha}{(u\log\theta_n)^\alpha\theta_n}\sum_{|y|\leq\theta_n}\mathbb{E}[g^\alpha_{B_n^2}(y)P(\min_{y'\sim y}\sigma(y')\leq\theta_n)]+o(1). \quad (3.4.39)$$

We show now that, up to a negligible error, we may substitute $P(\sigma(y)\leq\theta_n)$ for $P(\min_{y'\sim y}\sigma(y')\leq\theta_n)$ for all $y\in B_{\theta_n}$. To see this note for $y\in B_{\theta_n}$

$$P(\min_{y'\sim y}\sigma(y')\leq\theta_n)\leq P(\sigma(y)\leq\theta_n)+\textstyle\sum_{y'\sim y}P(\sigma(y')\leq\theta_n<\sigma(y)). \qquad (3.4.40)$$

Now, by the Markov property,

$$\begin{aligned}&\textstyle\sum_{y'\sim y}P(\sigma(y')\leq\theta_n<\sigma(y))\\ \leq\;&\textstyle\sum_{y'\sim y}P(\sigma(y')\leq h(\theta_n))P_{y'}(\sigma(y)>k(\theta_n))+P(\sigma(y')\in(h(\theta_n),\theta_n))\\ =\;&A_n^1(y)+A_n^2(y). \qquad (3.4.41)\end{aligned}$$

It thus suffices to establish that

$$\theta_n^{-1}(\log\theta_n)^{1-\alpha}\textstyle\sum_{|y|\leq\theta_n}\mathbb{E}[\widetilde{g}^\alpha_{B_n^1}(y)(A_n^1(y)+A_n^2(y))]=o(1). \qquad (3.4.42)$$

Lemma 3.3 in [31] states that there exists $c_9\in(0,\infty)$ such that, $\mathbb{P}$-a.s., for all $y\in B_{d_n(t)}$, $\widetilde{g}_{B_n^i}(y)\leq c_9\log\theta_n$ for $i=1,2$. Moreover, (3.3.12) of Lemma 3.12 tells us that $\sum_y A_n^2(y)\leq c_5\theta_n/(\log\theta_n)^2$. Hence the contribution of A_n^2 in the left hand side of (3.4.42) is of the order $o(1)$. To see that the same is true for A_n^1, we use (3.3.12) of Lemma 3.12 to bound

$$\mathbb{E}P(\sigma(y')\leq h(\theta_n))P_{y'}(\sigma(y)>k(\theta_n))\leq f_{h(\theta_n)}(|y'|)\mathbb{E}P(\sigma(x)>k(\theta_n)), \qquad (3.4.43)$$

where $|x|=1$. By recurrence and irreducibility, $\mathbb{E}P(\sigma(x)>k(\theta_n))\leq\varepsilon$ for N large enough and (3.4.43) implies that $\sum_y A_n^1(y)\leq c_5\varepsilon$. This concludes the proof of (3.4.42). Finally, combining (3.4.39)-(3.4.43),

$$\mathbb{E}\widetilde{\nu}_n^t(u,\infty)\leq\frac{t\log\theta_n\Gamma(1+\alpha)(1+\varepsilon)^\alpha}{(u\log\theta_n)^\alpha\theta_n}\sum_{|y|\leq\theta_n}\mathbb{E}[g^\alpha_{B_n^2}(y)P(\sigma(y)\leq\theta_n)]+o(1). \qquad (3.4.44)$$

We now show that (3.4.38) and (3.4.44) tend to the same limit $\mathcal{K}tu^{-\alpha}$. By Proposition 3.1 in [31] we know that there exists $\bar{K}$ such that, as $r\to\infty$, $(\bar{K}\log r)^{-1}\widetilde{g}_{B_r^{1/2}(0)}(0)$ converges $\mathbb{P}$-a.s. to one for $i=1,2$. Thus, $\mathbb{P}$-a.s.,

$$\lim_{N\to\infty}(\bar{K}\log\theta_n)^{-1}\widetilde{g}_{B_n^1}(0)=\lim_{N\to\infty}(\bar{K}\log\theta_n)^{-1}\widetilde{g}_{B_n^2}(0)=1. \qquad (3.4.45)$$

For $\varepsilon>0$ define $\mathcal{B}_n(y)\equiv\{|(\bar{K}\log\theta_n)^{-1}g_{B_n^2}(y)-1|\leq\varepsilon\}$. Then,

$$(3.4.44)\leq\tfrac{t\log\theta_n(1+\varepsilon)^\alpha\mathcal{K}'}{u^\alpha\theta_n}\textstyle\sum_{y\in\mathbb{Z}^d}\mathbb{E}[P(\sigma(y)\leq\theta_n)((1+\varepsilon)+c_9^\alpha/\mathcal{K}'\mathbb{1}_{\mathcal{B}_n^c(y)})], \qquad (3.4.46)$$

where $\mathcal{K}' \equiv \Gamma(1+\alpha)\bar{K}^{\alpha}$ and where we used that $\widetilde{g}_{B_n^1}(y) \le c_9 \log\theta_n$. Since Lemma 3.3 in [31] also states that there exists $c_8 \in (0,\infty)$ such that, $\mathbb{P}$-a.s., for all $y \in B_{d_n(t)}$, $\widetilde{g}_{B_n^1}(y) \ge c_8 \log\theta_n$, we can bound (3.4.38) from below in a similar way. Thus, the convergence of (3.4.38) and (3.4.44) follows if we can establish that

$$\lim_{N\to\infty} \theta_n (\log\theta_n)^{-1} \mathbb{E} E R_{\theta_n} = \bar{K}^{-1}, \tag{3.4.47}$$

$$\lim_{N\to\infty} \theta_n (\log\theta_n)^{-1} \sum_{y\in\mathbb{Z}^d} \mathbb{E}\left[P(\sigma(y) \le \theta_n) \mathbb{1}_{\mathcal{B}_n^c(y)})\right] = 0, \tag{3.4.48}$$

where R_{θ_n} is defined in (3.3.10). Let us first prove (3.4.48). By (3.3.12) of Lemma 3.12,

$$\sum_{y\in\mathbb{Z}^d} \mathbb{E}\left[P(\sigma(y) \le \theta_n) \mathbb{1}_{\mathcal{B}_n^c(y)})\right] \le \sum_{y\in\mathbb{Z}^d} f_{\theta_n}(|y|) \mathbb{P}(\mathcal{B}_n^c(y)) \le \tfrac{\theta_n}{\log\theta_n} \mathbb{P}(\mathcal{B}_n^c(0)), \tag{3.4.49}$$

where we used the identical distribution of the τ's. By Proposition 3.1 in [31] the probability of $\mathcal{B}_n^c(0)$ tends to zero, and so (3.4.48) holds. To prove (3.4.47), we construct upper and lower bounds for $\theta_n(\log\theta_n)^{-1}\mathbb{E} E R_{\theta_n}$ that coincide in the limit. We begin with the lower bound. By the Markov property,

$$\theta_n = \sum_{y\in\mathbb{Z}^d} E\widetilde{\ell}_{\theta_n}(y) \le \sum_{y\in\mathbb{Z}^d} P(\sigma(y) \le \theta_n) E_y \widetilde{\ell}_{\theta_n}(y). \tag{3.4.50}$$

To bound $\mathbb{E} E R_{\theta_n}$ from below it suffices to construct an upper bound on $E_y\widetilde{\ell}_{\theta_n}(y)$. We know from the proof of Lemma 3.12 that we may restrict the sum in the right hand side of (3.4.50) to $y \in B_{\theta_n}$. By Theorem 3.3.2 one can show that, $\mathbb{P}$-a.s., for all $y \in B_{d_n(t)}$, $E_y\widetilde{\ell}_{\theta_n}(y) \in (c_8\log\theta_n, c_9\log\theta_n)$, yielding

$$\theta_n \le \log\theta_n \sum_{|y|\le\theta_n} \mathbb{E}\left[P(\sigma(y) \le \theta_n > 0)\left(\bar{K}(1+\varepsilon) + c_8 \mathbb{1}_{\mathcal{B}_n^c(y)}\right)\right] + o(1). \tag{3.4.51}$$

Together with (3.4.48),

$$1 \le \theta_n^{-1} \log\theta_n \bar{K}(1+\varepsilon) \mathbb{E} E R_{\theta_n} + c_8 \mathbb{P}(\mathcal{B}_n^1), \tag{3.4.52}$$

i.e. $\lim_{N\to\infty} \theta_n^{-1} \log\theta_n \mathbb{E} E R_{\theta_n}$ is bounded below by $\bar{K}^{-1}$. For the upper bound we again use the Markov property and get that

$$\theta_n + k(\theta_n) = \sum_{y\in\mathbb{Z}^d} E\widetilde{\ell}_{\theta_n+k(\theta_n)}(y) \ge \sum_{y\in\mathbb{Z}^d} \mathbb{E}\left[P(\widetilde{\ell}_{\theta_n}(y) > 0) E_y \widetilde{\ell}_{k(\theta_n)}(y)\right]. \tag{3.4.53}$$

Since $k(\theta_n)\log\theta_n/\theta_n \to 0$, we can show that the upper bound coincides with the lower bound. The claim of (3.4.47) is proved. Finally, using (3.4.47) and (3.4.48) in (3.4.38) and (3.4.44),

$$\mathcal{K} u^{-\alpha} t (1-\varepsilon)^{1+\alpha} \le \lim_{N\to\infty} \mathbb{E}\widetilde{\nu}_n^t(u,\infty) \le \mathcal{K} u^{-\alpha} t (1+\varepsilon)^{1+\alpha}, \tag{3.4.54}$$

where $\mathcal{K} = \mathcal{K}'\bar{K}^{-1}$. Since $\varepsilon > 0$ is arbitrary this proves the convergence of $\mathbb{E}\widetilde{\nu}_n^t(u,\infty)$. This finishes the proof of Lemma 3.17 for $d = 2$. □

3.4.2 Bound on the variance of $\widetilde{\nu}_n^t(u,\infty)$

We construct a bound on the variance of $\widetilde{\nu}_n^t(u,\infty)$ which tends to zero fast enough.

Lemma 3.18. *For all $u > 0$, $t > 0$, for N large enough,*

$$\mathbb{E}(\widetilde{\nu}_n^t(u,\infty))^2 - (\mathbb{E}\widetilde{\nu}_n^t(u,\infty))^2 \le K(t) u^{-\alpha} \Big(\tfrac{(\log\log a_n)^3}{\log\theta_n} \mathbb{1}_{d=2} + \tfrac{(\log\theta_n)^2}{\sqrt{\theta_n}} \mathbb{1}_{d\ge 3} \Big), \tag{3.4.55}$$

where $K(t) \in (0,\infty)$ is independent of u.

Note that by (3.1.38), (3.1.39), and (3.3.16) the right hand side of (3.4.55) is summable.

The next lemma is designed to control $\mathbb{E}(\sum_{y\in\mathbb{Z}^d} Q_n^u(0,y))^2$ which arises in the proof of Lemma 3.18. To simplify notation we set

$$\widetilde{Q}_n^u(x) \equiv \sum_{y\in\mathbb{Z}^d} Q_n^u(x,y), \quad x\in\mathbb{Z}^d. \tag{3.4.56}$$

Lemma 3.19. *For all $u>0$ there exists $K\in(0,\infty)$ such that, for N large enough,*

$$\mathbb{E}(\widetilde{Q}_n^u(0))^2 \le \sum_{y:|y|\le\theta_n} \mathbb{E}(Q_n^u(0,y))^2 + c_n^{-3\alpha/2} \le \rho_n^u(d) \equiv Ku^{-\alpha}\rho_n(d), \tag{3.4.57}$$

where we set

$$\rho_n(d) \equiv \theta_n c_n^{-\alpha}(\log\theta_n)^{-2+\alpha}\mathbb{1}_{d=2} + \theta_n^{1/2}c_n^{-\alpha}\mathbb{1}_{d\ge3}. \tag{3.4.58}$$

The main step in the proof of Lemma 3.19 is to bound the sum over y of $\mathbb{E}(Q_n^u(0,y))^2$ and we first intuitively explain why this sum is smaller than $\rho_n(d)$. By (3.4.10) we know that $(Q_n^u(0,y))^2 \le c(\mathcal{P}_y(\ell_{B_n^2}(y)\gamma_n > u))^2(P(\sigma(y)\le\theta_n))^2 + o(1)$, where $c\in(0,\infty)$. In the proof of Lemma 3.19 we show that the first term is for every $y\in B_{\theta_n}$ of the order of $(c_n/\log\theta_n)^{-\alpha}\mathbb{1}_{d=2} + c_n^{-\alpha}\mathbb{1}_{d\ge3}$. The second term is the probability that y is visited by $\widetilde{J}$ and an independent copy $\widetilde{J}'$ during $[0,\theta_n]$ and we can use Lemma 3.12 to control the sum over all y of $\mathbb{E}(P(\sigma(y)\le\theta_n))^2$ by $\theta_n(\log\theta_n)^{-2}\mathbb{1}_{d=2} + \theta_n^{1/2}\mathbb{1}_{d\ge3}$. This explains the order of $\rho_n(d)$.

Proof of Lemma 3.19. By definition of $\widetilde{Q}_n^u(0)$ we have that

$$\mathbb{E}(\widetilde{Q}_n^u(0))^2 \le \sum_{y:|y|\le\theta_n} \mathbb{E}(Q_n^u(0,y))^2 + \sum_{\substack{y'\neq y\\ |y'-y|\le\theta_n}} \mathbb{E}(Q_n^u(0,y)Q_n^u(0,y')). \tag{3.4.59}$$

Let us first control the double sum in (3.4.59). Bounding $Q_n(0,y)\le \mathbb{1}_{y\in T_n}$, we get that

$$\sum_{\substack{y'\neq y\\ |y'-y|\le\theta_n}} \mathbb{E}(Q_n^u(0,y)Q_n^u(0,y')) \le \sum_{\substack{y'\neq y\\ |y'-y|\le\theta_n}} \mathbb{E}\mathbb{1}_{y,y'\in T_n} \le \frac{\theta_n^d}{(c_n\epsilon_n)^{2\alpha}} \le c_n^{-3\alpha/2}, \tag{3.4.60}$$

where we used (3.1.38) and (3.1.39). It remains to bound the first term on the right hand side of (3.4.59). For $y\in B_{\theta_n}$ we know by (3.4.9) and (3.4.10) that $\mathbb{E}(Q_n^u(0,y))^2$ is smaller than

$$\begin{aligned} &\mathbb{E}\left[\mathbb{1}_{y\in T_n}\left(P_y\left(\widetilde{\ell}_{\theta_n}(y)\gamma_n(y) > u\right)P(\sigma(y)\le\theta_n)\right)^2\right] \\ &\le \mathbb{E}\left[\mathbb{1}_{y\in T_n}\left(P_y(\widetilde{\ell}_{B_n^2}(y)\gamma_n(y)>u)P(\sigma(y)\le\theta_n)\right)^2\right] + e^{-c_4(\log\theta_n)^2}\mathbb{P}(y\in T_n). \end{aligned} \tag{3.4.61}$$

The contribution to $\mathbb{E}(\widetilde{Q}_n^u(x))^2$ of the second term in (3.4.61) is negligible because

$$\sum_{y:|y|\le\theta_n} \mathbb{P}(y\in T_n)e^{-c_4(\log\theta_n)^2} \le \theta_n^d c_n^{-\alpha}\epsilon_n^{-\alpha}e^{-c_4(\log\theta_n)^2} \ll \rho_n^u(d). \tag{3.4.62}$$

It remains to bound the first summand in (3.4.61). Recall from the proof of Lemma 3.17 that $\widetilde{\ell}_{B_n^2}(y)$ has exponential distribution with mean $g_{B_n^2}(y)\le c_8\log\theta_n\mathbb{1}_{d=2}+c_6\mathbb{1}_{d\ge3}$, $\mathbb{P}$-a.s for all $y\in B_{\theta_n}$, and that $P(\sigma(y)\le\theta_n)\le P(\min_{y'\sim y}\sigma(y)\le\theta_n)$. Thus, for all $y\in B_{\theta_n}$

$$\begin{aligned} &\mathbb{E}(\mathcal{P}(\widetilde{\ell}_{B_n^2}(y)\gamma_n(y)>u)\mathbb{1}_{y\in T_n})^2 \\ &\le \mathbb{E}\Big(P(\min_{y'\sim y}\sigma(y')\le\theta_n)\exp(-u(\gamma_n(y)(c_6\mathbb{1}_{d\ge3}+c_8\log\theta_n\mathbb{1}_{d=2}))^{-1})\Big)^2 \\ &\le \mathbb{E}\exp(-2u(\gamma_n(0)(c_6\mathbb{1}_{d\ge3}+c_8\log\theta_n\mathbb{1}_{d=2}))^{-1})\mathbb{E}(P(\min_{y'\sim y}\sigma(y')\le\theta_n))^2. \end{aligned} \tag{3.4.63}$$

We bound the terms in (3.4.63) separately. The expectation with respect to $\gamma_n(0)$ is, for some $C\in(0,\infty)$, bounded above by $Cu^{-\alpha}c_n^{-\alpha}(\mathbb{1}_{d\ge3}+\log\theta_n\mathbb{1}_{d=2})$. Moreover, by Lemma 3.12

$$\begin{aligned} \sum_{y\in\mathbb{Z}^d}\mathbb{E}(P(\min_{y'\sim y}\sigma(y')\le\theta_n))^2 &\le (2d)^2\sum_{y\in\mathbb{Z}^d}\mathbb{E}(P(\sigma(y)\le\theta_n))^2 \\ &\le c_5(\theta_n(\log\theta_n)^{-2}\mathbb{1}_{d=2}+\theta_n^{1/2}\mathbb{1}_{d\ge3}). \end{aligned} \tag{3.4.64}$$

Collecting (3.4.63)-(3.4.64) yields

$$\sum_{y\in B_{\theta_n}} \mathbb{E}\big(\mathcal{P}(\widetilde{\ell}_{B_n^2}(y)\gamma_n(y) > u)\mathbb{1}_{y\in T_n}\big)^2 \le K u^{-\alpha}\rho_n(d) = \rho_n^u(d), \tag{3.4.65}$$

for some $K \in (0,\infty)$. This finishes the proof of Lemma 3.19. □

We are now ready to present the proof of Lemma 3.18.

Proof of Lemma 3.18. The variance of $\widetilde{\nu}_n^t(u,\infty)$ is given by

$$k_n^2(t)\sum_{x\in B_{d_n(t)}}(\overline{\pi}_n^t(x))^2\big[\mathbb{E}\big(\widetilde{Q}_n^u(x)\big)^2 - \big(\mathbb{E}\widetilde{Q}_n^u(x)\big)^2\big] \tag{3.4.66}$$

$$+\; k_n^2(t)\sum_{x\neq x',|x-x'|\le 2\theta_n}\overline{\pi}_n^t(x)\overline{\pi}_n^t(x')\big[\mathbb{E}\big(\widetilde{Q}_n^u(x)\widetilde{Q}_n^u(x')\big) - \mathbb{E}\widetilde{Q}_n^u(x)\mathbb{E}\widetilde{Q}_n^u(x')\big], \tag{3.4.67}$$

where we used the fact that $\widetilde{Q}_n^u(x)$ only depends on $\tau(y)$ for $y \in B_{\theta_n}(x)$. Let us first bound (3.4.66). We begin with constructing bounds for $\overline{\pi}_n^t(x)$. As, $\mathbb{P}$-a.s., $U_x \le c_0(\log a_n)^3 \ll k\theta_n$, for all $x \in B_{d_n(t)}$, may apply (3.3.4) of Theorem 3.10 to get, $\mathbb{P}$-a.s., for every $x\in B_{d_n(t)}$,

$$\begin{aligned}\pi_n^t(x) \;&\le\; (k_n(t))^{-1}\Big(\sum_{k=\lfloor|x|/\theta_n\rfloor\vee 1}^{k_n(t)-1}(k\theta_n)^{-d/2}e^{-c_2|x|^2/(k\theta_n)} + \sum_{k=1}^{\lfloor|x|/\theta_n\rfloor\wedge 1}e^{-c_2|x|}\Big)\\ &\le\; c_1(k_n(t)\theta_n)^{-1}\int_{1/2|x|\vee\theta_n}^{a_nt} s^{-d/2}e^{-c_2|x|^2s^{-1}}ds,\end{aligned} \tag{3.4.68}$$

with the convention that $\sum_{k=1}^{0} = 0$. Let first $d=2$. An asymptotic analysis yields for $|x| \le 1/2\sqrt{a_n}$ that there exists $c' \in (0,\infty)$ such that

$$\pi_n^t(x) \le c'(k_n(t)\theta_n)^{-1}\log(a_n/|x|^2). \tag{3.4.69}$$

Moreover, when $|x| \gg \sqrt{a_n}$, we see that $\pi_n^t(x) \le e^{-c_2/2|x|^2/a_n}$. Using these bounds and the fact that the integral in (3.4.68) is decreasing in the norm of x, we know that there exists $c_3 \in (0,\infty)$ such that

$$\overline{\pi}_n^t(x) \le (k_n(t)\theta_n)^{-1}\begin{cases} c_3\log a_n, & \text{if } |x| \le \frac{\sqrt{a_n}}{(\log a_n)^2},\\ c_3\log\log a_n, & \text{if } \frac{\sqrt{a_n}}{(\log a_n)^2} \le |x| \le a_n^{1/2}\log\log a_n,\\ e^{-c_2/2(\log\log a_n)^2}, & \text{if } a_n^{1/2}\log\log a_n < |x|.\end{cases} \tag{3.4.70}$$

Now let $d\ge 3$. We substitute $u = c_2|x|^2s^{-1}$ in (3.4.68) and get

$$\pi_n^t(x) \le c''(k_n(t)\theta_n)^{-1}|x|^{2-d}\Gamma\big(d/2-2, |x|^2/a_n\big), \tag{3.4.71}$$

where $c'' \in (0,\infty)$. Thus, taking c_3 large enough,

$$\overline{\pi}_n^t(x) \le (k_n(t)\theta_n)^{-1}\begin{cases} c_3|x|^{2-d}, & \text{if } 0<|x|\le a_n^{1/2}\log\theta_n,\\ c_3|x|^{2-d}e^{-1/2(\log\theta_n)^2}, & \text{else.}\end{cases} \tag{3.4.72}$$

For $x=0$, we bound $\overline{\pi}_n^t(0) \equiv \overline{\pi}_n^t(y)$ for $|y|=1$. By (3.4.70) and (3.4.72),

$$\sum_{x\in B_{d_n(t)}}(k_n(t)\overline{\pi}_n^t(x))^2 \le \theta_n^{-2}\big(a_n\mathbb{1}_{d=2} + a_n^{1/2}\log\theta_n\mathbb{1}_{d\ge 3}\big). \tag{3.4.73}$$

Now we are ready to prove that (3.4.66) and (3.4.67) satisfy (3.4.55). We bound the variance in (3.4.66) by $\mathbb{E}(Q_n^u(0,y))^2$ and use Lemma 3.19 to obtain

$$(3.4.66) \le \sum_{x\in B_{d_n(t)}}(k_n(t)\overline{\pi}_n^t(x))^2\rho_n^u(d) \ll K(t)u^{-\alpha}\big(\tfrac{1}{\log\theta_n}\mathbb{1}_{d=2} + \tfrac{(\log\theta_n)^4}{\sqrt{\theta_n}\epsilon_n^\alpha}\mathbb{1}_{d\ge 3}\big), \tag{3.4.74}$$

for $K(t) \in (0, \infty)$. Since this satisfies (3.4.55), it suffices to control (3.4.67). We know that

$$(3.4.67) \leq (k_n(t))^2 \sum_{x,x':|x-x'|\leq 2\theta_n} \overline{\pi}_n^t(x)\overline{\pi}_n^t(x')\mathbb{E}\big(\widetilde{Q}_n^u(x)\widetilde{Q}_n^u(x')\big). \tag{3.4.75}$$

Fix $x, x' \in B_{d_n(t)}$. We proceed as in the proof of Lemma 3.19 to bound $\mathbb{E}\big(\widetilde{Q}_n^u(x)\widetilde{Q}_n^u(x')\big)$. Namely, by analogy to (3.4.60), (3.4.61) and (3.4.63), for some $K' \in (0, \infty)$,

$$\mathbb{E}\big(\widetilde{Q}_n^u(x)\widetilde{Q}_n^u(x')\big) \leq K'2du^{-\alpha}c_n^{-\alpha}\big((\log\theta_n)^\alpha \mathbb{1}_{d=2} + \mathbb{1}_{d\geq 3}\big)\mathbb{E}I_{\theta_n}(x,x') + c_n^{-3\alpha/2}, \tag{3.4.76}$$

where $I_{\theta_n}(x, x')$ is the expected intersection range of $\widetilde{J}$ starting in x and an independent copy $\widetilde{J}'$ starting in x' given by

$$I_{\theta_n}(x,x') \equiv \sum_{y:|x-y|\wedge|x'-y|\leq\theta_n} E_x E'_{x'} \mathbb{1}_{\sigma(y)\leq\theta_n}\mathbb{1}_{\sigma'(y)\leq\theta_n}. \tag{3.4.77}$$

Let us now distinguish two cases with respect to the size of $|x - x'|$ to deal with $I_{\theta_n}(x, x')$. For $x \in B_{d_n(t)}$ we define the sets $A_1(x) \equiv \{x' : \sqrt{\theta_n}\log\theta_n \leq |x - x'| \leq \theta_n\}$ and $A_2(x) \equiv B_{\theta_n}(x) \setminus A_1(x)$. Let $x' \in A_1(x)$. We bound $I_{\theta_n}(x, x')$ by

$$I_{\theta_n}(x,x') \leq P_x P'_{x'}(\exists y \in \mathbb{Z}^d : \max\{\sigma(y), \sigma'(y)\} \leq \theta_n) \max\{E_x R_{\theta_n}, E_{x'}R_{\theta_n}\}. \tag{3.4.78}$$

Since $x' \in A_1(x)$, the probability in (3.4.78) is bounded above by the probability that either $\widetilde{J}$ or $\widetilde{J}'$ go during $[0, \theta_n]$ further than distance $\frac{1}{2}\sqrt{\theta_n}\log\theta_n$ from their starting point. By (3.3.8) of Lemma 3.11 this is smaller than $e^{-c'(\log\theta_n)^2}$, where $c' = c_4/4$. Thus, by Lemma 3.12,

$$\mathbb{E}I_{\theta_n}(x,x') \leq c_5\theta_n \exp(-c'(\log\theta_n)^2). \tag{3.4.79}$$

We use (3.4.79) and get for $x \in B_{d_n(t)}, x' \in A_1(x)$

$$\mathbb{E}\big(\widetilde{Q}_n^u(x)\widetilde{Q}_n^u(x')\big) \leq c_5(\rho_n^u(d)\theta_n e^{-c'(\log\theta_n)^2} + c_n^{-3\alpha/2}). \tag{3.4.80}$$

By (3.4.70) and (3.4.72) we have for any $B_r(y)$ with $r \leq d_n(t)$ that

$$k_n(t)\sum_{x\in B_r(y)} \overline{\pi}_n^t(x) \leq \tfrac{\log\log a_n}{\theta_n}(\min(r^2, a_n)\mathbb{1}_{d\geq 3} + \min(r^2\log a_n, a_n)\mathbb{1}_{d=2}). \tag{3.4.81}$$

By (3.4.81),

$$(k_n(t))^2\sum_{x\in B_{d_n(t)},x'\in A_1(x)} \overline{\pi}_n^t(x)\overline{\pi}_n^t(x') \leq c_3 k_n(t)(\log\log a_n)^2. \tag{3.4.82}$$

Combining (3.4.80) and (3.4.82),

$$\sum_{x\in B_{d_n(t)},x'\in A_1(x)} \overline{\pi}_n^t(x)\overline{\pi}_n^t(x')\mathbb{E}\big(\widetilde{Q}_n^u(x)\widetilde{Q}_n^u(x')\big) \leq K(t)u^{-\alpha}e^{-c'(\log\theta_n)^2} + c_n^{-\alpha/2}, \tag{3.4.83}$$

which is smaller than the right hand side of (3.4.55). Let now $x' \in A_2(x)$. We distinguish whether $d \geq 3$ or $d = 2$. Let first $d \geq 3$. We bound by Cauchy Schwarz inequality $\mathbb{E}\big(\widetilde{Q}_n^u(x)\widetilde{Q}_n^u(x')\big) \leq \rho_n^u(d)$. By (3.4.81) we get

$$(k_n(t))^2\sum_{x\in B_{d_n(t)},x'\in A_2(x)} \overline{\pi}_n^t(x)\overline{\pi}_n^t(x')\rho_n^u(d) \leq c_3 k_n(t)\rho_n^u(d)(\log\theta_n)^2, \tag{3.4.84}$$

as desired in (3.4.55). This finishes the proof of Lemma 3.18 for $d \geq 3$.

Let $d = 2$. Fix $x \in B_{d_n(t)}$ and $x' \in A_2(x)$. We distinguish two further cases with respect to $|x - x'|$. For $k > 5c_4^{-1}$ we define the sets $B_1(x) \equiv A_2(x) \cap \{x' : |x - x'| \geq \sqrt{\theta_n k \log\log\theta_n}\}$ and $B_2(x) \equiv A_2(x) \setminus B_1(x)$. Let $x' \in B_1(x)$. As in (3.4.79), we get by Lemma 3.12,

$$\mathbb{E}I_{\theta_n}(x,x') \leq 16c_5\theta_n(\log\theta_n)^{-5}. \tag{3.4.85}$$

By (3.4.81) we have that

$$(k_n(t))^2 \sum_{x\in B_{d_n(t)},x'\in B_1(x)} \overline{\pi}_n^t(x)\overline{\pi}_n^t(x') \le c_3 k_n(t)\log\log a_n(\log\theta_n)^3. \qquad (3.4.86)$$

Together with (3.4.85),

$$\sum_{x\in B_{d_n(t)},x'\in B_1(x)} \overline{\pi}_n^t(x)\overline{\pi}_n^t(x')\mathbb{E}(\widetilde{Q}_n^u(x)\widetilde{Q}_n^u(x')) \le K'' u^{-\alpha} t\frac{(\log\log a_n)^3}{\log\theta_n}, \qquad (3.4.87)$$

as claimed in (3.4.55). Finally, let $x'\in B_2(x)$. By (3.4.81),

$$\sum_{x'\in B_2(x)} k_n(t)\overline{\pi}_n^t(x') \le \log\log a_n, \qquad (3.4.88)$$

and therefore by Cauchy Schwarz inequality

$$\sum_{x\in B_{d_n(t)}}\sum_{x'\in B_2(x)}(k_n(t))^2\overline{\pi}_n^t(x)\overline{\pi}_n^t(x')\mathbb{E}(\widetilde{Q}_n^u(x)\widetilde{Q}_n^u(x')) \le \frac{K(t)u^{-\alpha}(\log\log a_n)^3}{\log\theta_n}, \qquad (3.4.89)$$

which is as claimed in (3.4.55). This finishes the proof of Lemma 3.18. □

3.4.3 Convergence of $\mathbb{E}\widetilde{\sigma}_n^t(u,\infty)$

We establish that $\lim_{N\to\infty}\mathbb{E}\widetilde{\sigma}_n^t(u,\infty)=0$ and that the convergence speed is summable in N. By Lemma 3.19 we know that

$$\mathbb{E}(\widetilde{Q}_n^u(x))^2 \le 2\sum_y \mathbb{E}\left[(Q_n^u(x,y))^2\right] = 2\sum_{|y|\le\theta_n}\mathbb{E}\left[(Q_n^u(0,y))^2\right] \le 2\rho_n^u(d). \qquad (3.4.90)$$

Therefore, there exists $K\in(0,\infty)$ such that

$$\mathbb{E}\widetilde{\sigma}_n^t(u,\infty) \le 2k_n(t)\sum_{x\in B_{d_n(t)}}\overline{\pi}_n^t(x)\rho_n^u(d) \le Ku^{-\alpha}t\big(\tfrac{1}{\log\theta_n}\mathbb{1}_{d=2}+\tfrac{1}{\sqrt{\theta_n}}\mathbb{1}_{d\ge3}\big). \qquad (3.4.91)$$

Thus, $\lim_{N\to\infty}\mathbb{E}\widetilde{\sigma}_n^t(u,\infty)=0$.

3.4.4 Verification of Condition (C-4)

We follow the same strategy as in the verification of (C-2). We first prove for N large enough that $\mathbb{E}m_n^t(\varepsilon)\le C(t)\varepsilon^{1-\alpha}$. Then we establish that the variance of $m_n^t(\varepsilon)$ is summable in N. Since this is similar to the proof of Lemma 3.18, we only indicate the needed changes at the end of this section.

Let us bound $\mathbb{E}m_n^t(\varepsilon)$. Since the τ's are i.i.d., it suffices to find $c\in(0,\infty)$ such that

$$\sum_{y\in B_{\theta_n}}\mathbb{E}\left[M_n^\varepsilon(0,y)\right] \le c\theta_n a_n^{-1}\varepsilon^{1-\alpha}. \qquad (3.4.92)$$

Fix $y\in B_{\theta_n}$ and set $h(\theta_n)=\theta_n-k(\theta_n)$ for $k(\theta_n)=\theta_n^{3/4}$. As in (3.4.3) and (3.4.9), by the Markov property,

$$\begin{aligned} M_n^\varepsilon(0,y) &\le \varepsilon P(\sigma(y)\in(h(\theta_n),\theta_n))\mathbb{1}_{y\in T_n} \\ &\quad+ P(\sigma(y)\le h(\theta_n))\gamma_n(y)\mathcal{E}_y\big(\widetilde{\ell}_{\theta_n}(y)\mathbb{1}_{\widetilde{\ell}_{k(\theta_n)}(y)\gamma_n(y)\le\varepsilon}\big)\mathbb{1}_{y\in T_n} \\ &\equiv M_{n,1}(y)+M_{n,2}(y). \end{aligned} \qquad (3.4.93)$$

Let us first establish, that $\mathbb{P}$-a.s. the sum over $M_{n,1}(y)$ tends to zero. Following the same argumentation as between (3.4.12) and (3.4.15), we can show that

$$M_{n,1}(y) \le \varepsilon(P(\min_{y'\sim y}\sigma(y)\in(h(\theta_n),\theta_n))+c_n^{-\theta/2})\mathbb{1}_{y\in T_n}. \qquad (3.4.94)$$

Since $\min_{y'\sim y}\sigma(y)$ is independent of $\gamma_n(y)$, we get by Lemma 3.12 that

$$\textstyle\sum_y \mathbb{E}M_{n,1}(y) \le \sum_y \mathbb{E}\big[P(\sigma(y)\in(h(\theta_n),\theta_n))\mathbb{1}_{y\in T_n}\big] \le c_5 c_n^{-\alpha}\epsilon_n^{-\alpha}\theta_n^{-3/4}. \tag{3.4.95}$$

By a first order Chebyshev inequality we conclude that the sum over $M_{n,1}(y)$ tends $\mathbb{P}$-a.s. to zero. Let us now bound the expectation of $M_{n,2}(y)$. First we calculate the expected value with respect to $\mathcal{E}_y$ in $M_{n,2}(y)$. As in (3.4.10) and (3.4.11), we can show that, up to an error of the order of $e^{-c_4(\log\theta_n)^2}$, we can bound for all $y\in B_{\theta_n}\cap T_n$, $\widetilde{\ell}_{\eta(B_n^1)}(y)\le \widetilde{\ell}_{k(\theta_n)}(y)$ and $\widetilde{\ell}_{\theta_n}(y)\le\widetilde{\ell}_{\eta(B_n^2)}(y)$, where $B_n^1=B_{k(\theta_n)^{1/2}(\log\theta_n)^{-2}}(y)$ and $B_n^2=B_{\theta_n^{1/2}\log\theta_n}(y)$. Setting $A(y)\equiv\{\widetilde{\ell}_{\eta(B_n^1)}(y)\gamma_n(y)\le\varepsilon\}$, we get

$$\begin{aligned}\mathcal{E}_y\widetilde{\ell}_{\theta_n}(y)\mathbb{1}_{\widetilde{\ell}_{k(\theta_n)}(y)\gamma_n(y)\le\varepsilon} &\le \mathcal{E}_y\mathbb{1}_{A(y)}\widetilde{\ell}_{\eta(B_n^2)}(y)\\ &= \big(\mathcal{E}_y\mathbb{1}_{A(y)}\widetilde{\ell}_{\eta(B_n^1)}(y)+\mathcal{E}_y\mathbb{1}_{A(y)}\big(\widetilde{\ell}_{\eta(B_n^2)}(y)-\widetilde{\ell}_{\eta(B_n^1)}(y)\big)\big).\end{aligned} \tag{3.4.96}$$

By the strong Markov property, the second term in (3.4.96) is given by

$$\textstyle\sum_{z\in\partial B_n^1}\mathcal{E}_y\big(\mathbb{1}_{A(y)}\mathbb{1}_{\widetilde{J}(\eta(B_n^1))=z}\big)E_z\int_0^{\eta(B_n^2)}\mathbb{1}_{\widetilde{J}(s)=y}ds \le g_{B_n^2}(y)\mathcal{P}_y(A(y)), \tag{3.4.97}$$

where we used $E_z\int_0^{\eta(B_n^2)}\mathbb{1}_{\widetilde{J}(s)=y}ds\le g_{B_n^2}(y)$. The first term in (3.4.96) equals

$$g_{B_n^1}(y)\big[1-\exp(-\varepsilon/(\gamma_n(y)g_{B_n^1}(y)))\big]=g_{B_n^1}(y)\mathcal{P}_y(A(y)). \tag{3.4.98}$$

Using (3.4.97) and (3.4.98) and the fact that $g_{B_n^1}(y)\le g_{B_n^2}(y)$, (3.4.96) is bounded by

$$\begin{aligned}&2g_{B_n^2}(y)\big[1-\exp(-\varepsilon/(\gamma_n(y)g_{B_n^1}(y)))\big]\\ &\le 2c_8/c_7\bar{g}_n^d(y)\big[1-\exp(-\varepsilon/(\gamma_n(y)\bar{g}_n^d(y)))\big],\end{aligned} \tag{3.4.99}$$

where $\bar{g}_n^d(y)\equiv c_7(\log\theta_n\mathbb{1}_{d=2}+g_\infty(y)\mathbb{1}_{d\ge3})$ and where we used (3.4.21) (for $d\ge3$) and Lemma 3.3 in [31] (for $d=2$). Together with (3.4.12), we get

$$\mathbb{E}M_{n,2}(y)\le\varepsilon\,\mathbb{E}\big[P(\min_{y'\sim y}\sigma(y')\le\theta_n)\bar{g}_n^d(y)\gamma_n(y)(1-e^{-\varepsilon(\gamma_n(y)\bar{g}_n^d(y))^{-1}})\mathbb{1}_{y\in T_n}\big]. \tag{3.4.100}$$

An asymptotic analysis and the definition of $\bar{g}_n^d(y)\le c_7(\log\theta_n\mathbb{1}_{d=2}+c_6\mathbb{1}_{d\ge3})$ yield

$$\begin{aligned}(3.4.100)\quad &\le\quad c'\,\varepsilon\,\mathbb{E}\Big[P(\min_{y'\sim y}\sigma(y')\le\theta_n)e^{-2\varepsilon(c'\gamma_n(y)\bar{g}_n^d(y))^{-1}}\mathbb{1}_{y\in T_n}\Big]\\ &\le\quad c''\,(\log\theta_n\mathbb{1}_{d=2}+c_6\mathbb{1}_{d\ge3})^\alpha c_n^{-\alpha}\varepsilon^{1-\alpha}\mathbb{E}P(\min_{y'\sim y}\sigma(y')\le\theta_n).\end{aligned} \tag{3.4.101}$$

for some $c',c''\in(0,\infty)$. By Lemma 3.12 the sum over all y of $\mathbb{E}P(\min_{y'\sim y}\sigma(y')\le\theta_n)$ is bounded above by $c_5\theta_n((\log\theta_n)^{-1}\mathbb{1}_{d=2}+\mathbb{1}_{d\ge3})$, so

$$\textstyle\sum_y\mathbb{E}M_{n,2}(y)\le c''c_6c_5t\varepsilon^{1-\alpha}, \tag{3.4.102}$$

i.e. (3.4.92) is satisfied. Thus, $\lim_{N\to\infty}\mathbb{E}m_n^t(\varepsilon)\le c\varepsilon^{1-\alpha}$.

Finally, let us explain how one can show that $m_n^t(\varepsilon)$ concentrates around its mean. By (3.4.95) and a first order Chebyshev inequality, $\mathbb{P}$-a.s., the contribution of $\sum_y M_{n,1}(y)$ to $m_n^t(\varepsilon)$ is negligible. It remains to establish that $\sum_y M_{n,2}(y)$ concentrates around its mean. But $M_{n,2}(y)$ is of the same form as $\widetilde{Q}_n^u(y)$ and we can prove the result of Lemma 3.18 for $m_n^t(\varepsilon)$ as well. This finishes the verification of (C-4).

3.4.5 Verification of Condition (C-5)

We proceed as in the verification of (C-2) and (C-4) to establish that (C-5) is satisfied. Namely, we first take the expected value in the left hand side of (3.3.35) and (3.3.36) and prove that both are bounded above by $C(u,t)\varepsilon$ for some $C(u,t) \in (0,\infty)$. Then, we establish that the variance of both left hand sides is summable in N. Since the proofs are similar, we only prove the claim for (3.3.35). The expectation of the left hand side of (3.3.35) is given by

$$\textstyle\sum_{(x,k)\in\mathcal{A}_n}(k\theta_n)^{-d/2}e^{-c_2|x|^2/k\theta_n}\sum_y \mathbb{E}Q_n^u(x,y). \tag{3.4.103}$$

By (3.4.2) the second sum in (3.4.103) is, for N large enough, smaller than $2\nu(u,\infty)\theta_n/a_n$, and so

$$(3.4.103) \le 2\nu(u,\infty)a_n^{-1}\theta_n\sum_{k=1}^{k_n(t)}(k\theta_n)^{-d/2}\sum_{|x|^2<\varepsilon k\theta_n \,\vee\, |x|^2>k\theta_n/\varepsilon} e^{-c_2|x|^2/k\theta_n}. \tag{3.4.104}$$

Let us first control the contribution of $x \in B_{\sqrt{\varepsilon k\theta_n}}$. Bounding the exponential term by one and using the fact that $|\{x : |x|^2 < \varepsilon k\theta_n\}| \le C\varepsilon^{d/2}(k\theta_n)^{d/2}$ for some $C \in (0,\infty)$, we get

$$\textstyle 2\nu(u,\infty)a_n^{-1}\theta_n\sum_{k=1}^{k_n(t)}(k\theta_n)^{-d/2}\sum_{|x|^2<\varepsilon k\theta_n}e^{-c_2|x|^2/k\theta_n} \le K(u,t)\varepsilon^{d/2}, \tag{3.4.105}$$

where $K(u,t) \in (0,\infty)$, as desired. The contribution of $x \in B_{d_n(t)} \setminus B_{(k\theta_n/\varepsilon)^{1/2}}$ satisfies

$$\textstyle\sum_{|x|^2>k\theta_n/\varepsilon}(k\theta_n)^{-d/2}e^{-c_2|x|^2/k\theta_n} \le C\int_{1/\varepsilon}^{\infty}y^{-d}e^{-c_2y} \le Ce^{-c_2/\varepsilon}. \tag{3.4.106}$$

Combining (3.4.105) and (3.4.106), we see that (3.4.103) is bounded above by $K(u,t)\varepsilon^{d/2}$.

To show that the variance of the left hand side of (3.3.35) is summable in N, we can use the same calculations as in Lemma 3.18. Doing this, we get that it is bounded above by $\rho_n^u(d)$. This finishes the verification of (C-5).

3.4.6 Conclusion of the proof

We are now ready to conclude the proof of Theorem 3.4. By Lemma 3.17, for all $u > 0$, $t > 0$, $\mathbb{E}\tilde{\nu}_n^t(u,\infty) \to t\nu(u,\infty)$ as $N \to \infty$. Together with Lemma 3.18 this shows that (C-2) is satisfied. By the results of Section 3.4.3, $\mathbb{E}\tilde{\sigma}_n^t(u,\infty)$ tends to zero and is summable in N. This and a first order Chebyshev inequality yield (C-3). By the same arguments as those used to establish (C-2), (C-4) follows from the results of Section 3.4.4. Finally, it follows from the results of Section 3.4.5 that (C-5) is satisfied. Hence, we may apply Proposition 3.15 and get that $\overline{S}_n^{\widetilde{J},b} \overset{J_1}{\Longrightarrow} V_\alpha$. By Lemma 3.14 this proves that $S_n^{\widetilde{J},b} \overset{J_1}{\Longrightarrow} V_\alpha$, as claimed in Theorem 3.4.

3.5 Application to dynamics of trap models

In this section we prove Theorem 3.7 and Theorem 3.8. We verify the conditions of Theorem 3.1 for more general dynamics than simple random walk of BTM. The proof is divided into several steps, which we now explain. The first step is as in the application to BATM, namely we introduce longer subsequences. In the next step, we introduce a new set of conditions, (D-2)-(D-4) that imply (A-2)-(A-4) in this setting and show that (A-0) and (A-1) hold. The verifications of (D-2)-(D-4) are contained in Sections 3.5.1-3.5.3. Finally, we conclude the proof of Theorem 3.7 in Section 3.5.4.

In order to prove Theorem 3.7, i.e. to obtain $\mathbb{P}$-a.s. convergence on time scales $c_n = n$, we consider longer subsequences. Namely, by the same reasoning as in Section 3.3.2 we assume from now on that $c_n = \exp(N^k)$, where $k = 7\gamma_3/(1-\alpha)$, and consider the limit as $N \to \infty$.

We now introduce a new set of conditions, (D-2)-(D-4) which imply (A-2)-(A-4) for the dynamics of BTM. To this end we use the same notation as in the previous sections, namely we write $d_n(t) \equiv \lfloor a_n t\rfloor^{1/2} \log\lfloor a_n t\rfloor$ and $B_{d_n(t)} \equiv \left\{x \in \mathbb{Z}^d : |x| \le d_n(t)\right\}$, where $|\cdot|$ denotes the Euclidian norm on $\mathbb{Z}^d$. Our quantities of interest are for $u > 0$, $t > 0$, and $\varepsilon > 0$ given by

$$\overline{\nu}_n^t(u,\infty) \equiv k_n(t) \textstyle\sum_{x \in B_{d_n(t)}} \pi_n^t(x) Q_n^u(x), \tag{3.5.1}$$

$$\overline{\sigma}_n^t(u,\infty) \equiv k_n(t) \textstyle\sum_{x \in B_{d_n(t)}} \pi_n^t(x) (Q_n^u(x))^2 \tag{3.5.2}$$

$$\overline{m}_n^t(u,\infty) \equiv k_n(t) \textstyle\sum_{x \in B_{d_n(t)}} \pi_n^t(x) c_n^{-1} \mathcal{E}_x(Z_{n,1}^J \mathbb{1}_{Z_{n,1}^J \le c_n\varepsilon}). \tag{3.5.3}$$

We are now ready to state (D-2)-(D-4). They are formulated for fixed $\omega \in \Omega$ and fixed $u > 0$, $t > 0$, and $\varepsilon > 0$.

(D-2) $\lim_{N\to\infty} \overline{\nu}_n^t(u,\infty) = t\nu(u,\infty)$.

(D-3) $\lim_{N\to\infty} \overline{\sigma}_n^t(u,\infty) = 0$.

(D-4) There exists $K' \in (0,\infty)$ such that

$$\lim_{N\to\infty} \overline{m}_n^t(\varepsilon) \le K' t \varepsilon^{1-\alpha}. \tag{3.5.4}$$

Proposition 3.20. *Suppose that (D-2)-(D-4) are $\mathbb{P}$-a.s. satisfied for all $u > 0$, $t > 0$, and $\varepsilon > 0$. Then we have, $\mathbb{P}$-a.s., $S_n^{J,b} \stackrel{J_1}{\Longrightarrow} V_\nu$.*

Proof. We show that (D-2)-(D-4) $\Rightarrow$ (A-2)-(A-4) and verify (A-0) and (A-1). Since ν is continuous, we know by Lemma 3.9 that it suffices to show that (A-0)-(A-4) hold $\mathbb{P}$-a.s. for fixed $u > 0$, $t > 0$, and $\varepsilon > 0$. Let $u > 0$, $t > 0$, and $\varepsilon > 0$. First we prove that (D-2)-(D-4) $\Rightarrow$ (A-2)-(A-4). The main difference between (D-2)-(D-4) and (A-2)-(A-4) is that in the latter we are summing over all $x \in \mathbb{Z}^d$, whereas in the first we restrict the summation to $x \in B_{d_n(t)}$. Moreover, in (D-4) we require that $\overline{m}_n^t(\varepsilon)$ tends to zero as $\varepsilon \to 0$ with a certain speed, whereas in (A-4) we only ask that it tends to zero. Therefore, bounding $Q_n^u(x) \le 1$, respectively $c_n^{-1}\mathcal{E}_x(Z_{n,1}^J \mathbb{1}_{Z_{n,1}^J \le c_n\varepsilon}) \le \varepsilon$, it is sufficient to show that

$$\lim_{N\to\infty} k_n(t) \textstyle\sum_{x \notin B_{d_n(t)}} \pi_n^t(x) = 0. \tag{3.5.5}$$

For fixed N we can rewrite the sum in (3.5.5) as follows

$$\begin{aligned} k_n(t) \textstyle\sum_{x \notin B_{d_n(t)}} \pi_n^t(x) &= \textstyle\sum_{k=1}^{k_n(t)} P(J(k\theta_n) \notin B_{d_n(t)}) \\ &\le \textstyle\sum_{k=\theta_n}^{\lfloor a_n t\rfloor} P(J(k) \notin B_{d_n(t)}) \\ &\le \textstyle\sum_{k=\theta_n}^{\lfloor a_n t\rfloor} P(J(k) \notin B_{\sqrt{k}\log k}). \end{aligned} \tag{3.5.6}$$

We construct now a bound for the probability that J exits in k steps the ball $B_{\sqrt{k}\log k}$, where $k = \theta_n, \dots, \lfloor a_n t\rfloor$. Let Γ be the covariance matrix of π_Y. Since Y_1 has mean zero, and since Γ is finite and nonsingular, the mapping $\mathcal{I} : \mathbb{R}^d \to (0,\infty)$ given by

$$\mathcal{I}(x) = d^{-1/2} |x \cdot \Gamma^{-1} x|, \quad x \in \mathbb{R}^d, \tag{3.5.7}$$

is a norm and there exists δ' such that $\delta'\mathcal{I}(x) \le |x| \le \delta'^{-1}\mathcal{I}(x)$. By the assumption that the moment generating function of Y_1 exists for all $|t| \le \delta$, where $\delta > 0$, we deduce from Corollary 12.2.7 in [47] that there exists $c_1', c_2 > 0$ such that we have for $0 < s \le \delta\sqrt{k}/2$

$$P(\max_{j=1,\dots,k} |J(j)| > s\sqrt{k}) \le c_2 e^{-c_1'\delta'^2 s^2} = c_2 e^{-c_1 s^2}, \tag{3.5.8}$$

where $c_1 \equiv c_1' \delta'^2$. For N large we may use (3.5.8) for $s = \log k \leq \delta\sqrt{k}/2$, to find that

$$(3.5.6) \leq \sum_{k=\theta_n}^{\lfloor a_n t \rfloor} c_2 e^{-c_1 \delta^2 (\log k)^2}, \tag{3.5.9}$$

which vanishes as $N \to \infty$. This proves (3.5.5) and hence (D-2)-(D-4) $\Rightarrow$ (A-2)-(A-4).

Let us verify (A-0), that is we show that, $\mathbb{P}$-a.s., $c_n^{-1}\sigma^J$ vanishes. Since J has initial distribution δ_0, we have for $\varepsilon > 0$

$$\mathcal{P}(c_n^{-1}\sigma^J > \varepsilon) = \mathcal{P}\left(c_n^{-1}\tau(0)e_1 > \varepsilon\right) = \exp\left(-c_n\varepsilon\tau(0)^{-1}\right). \tag{3.5.10}$$

Taking expectation with respect to the random environment we have for $\bar{C} \in (0, \infty)$ that

$$\mathbb{E}\exp\left(-c_n\varepsilon/\tau(0)\right) = \int_0^\infty e^{-y}\mathbb{P}\left(\tau(0) > c_n\varepsilon y^{-1}\right) dy \leq \bar{C} c_n^{-\alpha}\varepsilon^{-\alpha}\Gamma(1-\alpha). \tag{3.5.11}$$

By a first order Chebyshev inequality and Borel Cantelli Lemma this proves (A-0).

We show now that (A-1) holds, i.e. that for all $t > 0$ there exists $c < \infty$ such that, uniformly in $x \in \mathcal{V}$,

$$\lim_{n\to\infty} \sum_{k=1}^{k_n(t)-1} P(J(k\theta_n) = x) = 0, \tag{3.5.12}$$

and

$$\lim_{n\to\infty} \sum_{k=1}^{k_n(t)-1} P_x(J(k\theta_n) = x) < c. \tag{3.5.13}$$

By the assumptions that π_Y is symmetric and has finite covariance matrix, Theorem 2.3.9 and Theorem 2.3.10 in [47] imply that there exist $c_3, c_4 \in (0, \infty)$ such that for all $x \in \mathbb{Z}^d$ and $k \geq 1$ we have

$$P(J(k) = x) \leq c_3 k^{-d/2} \exp(-c_4|x|^2/k). \tag{3.5.14}$$

In particular we have that

$$\sup_{x\in\mathbb{Z}^d} P(J(k) = x) \leq c_3 k^{-d/2}. \tag{3.5.15}$$

This is increasing in d and we get that

$$\sum_{k=1}^{k_n(t)-1} P(J(k\theta_n) = x) \leq c_3\theta_n^{-1} \sum_{k=1}^{k_n(t)-1} k^{-1} \leq c_3\theta_n^{-1}\log a_n, \tag{3.5.16}$$

which by construction of a_n and θ_n implies (3.5.12). Moreover, since we have by construction of J that $P_x(J(k) = x) = P(J(k) = 0)$, (3.5.13) follows from (3.5.16). This finishes the verification of (A-1). The proof of Proposition 3.20 is complete. □

3.5.1 Verification of (D-2)

We show now that (D-2) is satisfied. We do not show the convergence of $\overline{\nu}_n^t$, but instead of its Laplace transform. Namely, for $v > 0$, we set

$$\hat{\nu}_n^t(v) \equiv \sum_{x\in B_{d_n(t)}} k_n(t) \int_0^\infty \pi_n(t) Q_n^x(u) e^{-uv} du. \tag{3.5.17}$$

The proof that for all $v > 0$, $\hat{\nu}_n^t(v)$ converges, $\mathbb{P}$-a.s., to the Laplace transform of $t\nu(u,\infty)$ is divided into two steps. First we show that the expectation of $\hat{\nu}_n^t(v)$ converges to the Laplace transform of $t\nu(u,\infty)$. Then, we prove by a second order Chebyshev inequality that, $\mathbb{P}$-a.s., $\hat{\nu}_n^t(v)$ concentrates around its mean value.

Convergence of $\mathbb{E}\hat{\nu}_n^t(v)$

Let $v > 0$ and $t > 0$. By a substitution we have that

$$\begin{aligned}\mathbb{E}\hat{\nu}_n^t(v) &= k_n(t)v^{-1}\left(\sum_{x\in B_{d_n(t)}}\pi_n^t(x)\left(1-\mathbb{E}\mathcal{E}_x e^{-vZ_n}\right)\right)\\ &\sim k_n(t)v^{-1}\left(1-\mathbb{E}\mathcal{E}e^{-vZ_n}\right),\end{aligned}\tag{3.5.18}$$

where we used the identical distribution of the τ's and the fact that $k_n(t)\sum_{x\in B_{d_n(t)}}\pi_n^t(x)\sim k_n(t)$. Let $n_s(J)$ denote the number of points that J visits exactly s times. By partial integration and by the assumption on the distribution of the τ's we have that

$$\begin{aligned}\left(1-\mathbb{E}\mathcal{E}e^{-vZ_n}\right) &= 1-\sum_{\{n_s\}}P(\{n_s\})\prod_{s=1}^{\theta_n}\left(\mathbb{E}\left(\tfrac{1}{1+\tau(0)v/c_n}\right)^s\right)^{n_s}\\ &= \sum_{\{n_s\}}P(\{n_s\})\left(1-\prod_{s=1}^{\theta_n}\left(\mathbb{E}\left(\tfrac{1}{1+\tau(0)v/c_n}\right)^s\right)^{n_s}\right)\\ &= \sum_{\{n_s\}}P(\{n_s\})\left(1-\prod_{s=1}^{\theta_n}\left(1-c_n^{-\alpha}v^\alpha\tfrac{\Gamma(s+\alpha)\Gamma(1-\alpha)}{\Gamma(s)}\right)^{n_s}\right)\\ &= \sum_{\{n_s\}}P(\{n_s\})\sum_{s=1}^{\theta_n}n_s c_n^{-\alpha}v^\alpha\tfrac{\Gamma(s+\alpha)\Gamma(1-\alpha)}{\Gamma(s)}+O(\theta_n c_n^{-2\alpha}),\end{aligned}\tag{3.5.19}$$

where one sees that the contribution of $O(\theta_n c_n^{-2\alpha})$ to the right hand side of (3.5.18) is negligible. Let us further rewrite the first term in the right hand side of (3.5.19). Writing $\ell_{\theta_n}(x)$ for the number of visits of J to x during the first θ_n steps we have

$$(3.5.19)=c_n^{-\alpha}v^\alpha\Gamma(1-\alpha)\sum_{x\in\mathbb{Z}^d}E\left(\tfrac{\Gamma(\ell_{\theta_n}(x)+\alpha)}{\Gamma(\ell_{\theta_n}(x))}\right)+O(\theta_n c_n^{-2\alpha}).\tag{3.5.20}$$

We distinguish now two different cases w.r.t. the dimension. Let first $d\geq 3$. Then, by (3.5.15), J is transient. We use Proposition 2.1 from [7] to calculate the expected value of the sum over $\frac{\Gamma(\ell_{\theta_n}(x)+\alpha)}{\Gamma(\ell_{\theta_n}(x))}$. This proposition states for $\beta\in[0,\infty)$ that

$$\lim_{N\to\infty}\theta_n^{-1}\sum_{x\in\mathbb{Z}^d}E\ell_{\theta_n}^\beta(x)=\sum_{j\geq 1}j^\beta\gamma^2(1-\gamma)^{j-1},\tag{3.5.21}$$

where $\gamma\equiv P(J(k)\neq 0,\ \forall k\geq 1)$ is the probability of no return to 0. Notice that $\gamma\in(0,1)$. The proof of (3.5.21) does not use that inside the expectation we are taking polynomials of the local time and therefore one can show as in the proof of Proposition 2.1 in [7] that

$$\lim_{N\to\infty}\theta_n^{-1}\sum_{x\in\mathbb{Z}^d}E\left(\tfrac{\Gamma(\ell_{\theta_n}(x)+\alpha)}{\Gamma(\ell_{\theta_n}(x))}\right)=\sum_{j\geq 1}\tfrac{\Gamma(j+\alpha)}{\Gamma(j)}\gamma^2(1-\gamma)^{j-1}\equiv\mathcal{K}'.\tag{3.5.22}$$

Note that $\mathcal{K}'=\mathcal{K}'(d,\alpha,\pi_Y)<\infty$, for all $\alpha\in(0,1)$ because $\gamma\in(0,1)$. Together with (3.5.19), we see that, for $d\geq 3$,

$$\lim_{N\to\infty}\mathbb{E}\hat{\nu}_n^t(v)=tv^{\alpha-1}\mathcal{K}'\Gamma(1-\alpha).\tag{3.5.23}$$

Let $d=2$. We use Theorem 1 in [30], which states for all $\beta\in[0,\infty)$, P-a.s.,

$$\lim_{N\to\infty}\theta_n^{-1}(\log\theta_n)^{1-\beta}\sum_{x\in\mathbb{Z}^d}\ell_{\theta_n}^\beta(x)=\mathcal{K}',\tag{3.5.24}$$

for $\mathcal{K}'=\mathcal{K}'(d,\alpha,\pi_Y)<\infty$. In fact, since $\mathcal{K}'$ is a multiple of the β^{th} moment of the exponential distribution, one can extend this result to $\beta>-1$. The proof of this theorem uses the fact that the sum is over polynomial powers of the local times and therefore we cannot proceed as in $d\geq 3$. Instead, we rewrite (3.5.20) as follows

$$\tfrac{\Gamma(1-\alpha)v^\alpha}{c_n^\alpha}\left(\sum_{x\in\mathbb{Z}^d}E\ell_{\theta_n}(x)^\alpha+\sum_{x\in\mathbb{Z}^d}E\left[\left(\tfrac{\Gamma(\ell_{\theta_n}(x)+\alpha)}{\Gamma(\ell_{\theta_n}(x))}-\ell_{\theta_n}(x)^\alpha\right)\mathbb{1}_{\ell_{\theta_n}(x)>0}\right]\right).\tag{3.5.25}$$

By Theorem 1 from [30], we know that

$$\lim_{N\to\infty} k_n(t)c_n^{-\alpha}v^{\alpha-1}\Gamma(1-\alpha)E\sum_{x\in\mathbb{Z}^d}\ell_{\theta_n}(x)^\alpha = tv^{\alpha-1}\mathcal{K}'\Gamma(1-\alpha), \tag{3.5.26}$$

where we use the fact that $a_n c_n^{-\alpha} = (\log\theta_n)^{1-\alpha}$. It remains to show that the second term in (3.5.25) is of smaller order than $k_n(t)$. By Stirling's formula we bound for $x \in \mathbb{Z}^d$ such that $\ell_{\theta_n}(x) > 0$

$$\frac{\Gamma(\ell_{\theta_n}(x)+\alpha)}{\Gamma(\ell_{\theta_n}(x))} \leq \ell_{\theta_n}(x)^\alpha \left(\frac{\ell_{\theta_n}(x)}{\ell_{\theta_n}(x)+\alpha}\right)^{1/2}\left(1+\frac{\alpha}{\ell_{\theta_n}(x)}\right)^{\ell_{\theta_n}(x)+\alpha} e^{\mu(\ell_{\theta_n}(x)+\alpha)}, \tag{3.5.27}$$

for a function μ such that $\mu(t) \in (0, 1/(12t))$ for $t > 0$. In particular, the second term in (3.5.25) is bounded above by

$$v^\alpha c_n^{-\alpha}\Gamma(1-\alpha)\sum_{x\in\mathbb{Z}^d}E\Big[\ell_{\theta_n}(x)^\alpha \mathbb{1}_{\ell_{\theta_n}(x)>0}\left(e^{1/(12(\ell_{\theta_n}(x)+\alpha))}-1\right)\Big] \tag{3.5.28}$$

$$\leq \quad v^\alpha c_n^{-\alpha}\Gamma(1-\alpha)\sum_{x\in\mathbb{Z}^d}E\Big[\ell_{\theta_n}(x)^{\alpha-1}\mathbb{1}_{\ell_{\theta_n}(x)>0}\Big]. \tag{3.5.29}$$

We use Theorem 1 in [30] for $\beta = -1+\alpha > -1$ to see that

$$\text{(3.5.29)} \sim \theta_n c_n^{-\alpha}(\theta_n\log\theta_n)^{\alpha-2} = k_n(t)(\log\theta_n)^{-1}, \tag{3.5.30}$$

which is of smaller order than $k_n(t)$. Thus, (3.5.29) does not contribute to $\lim_{N\to\infty}\mathbb{E}\hat{\nu}_n^t(v)$ and we find by (3.5.26) that $\lim_{N\to\infty}\mathbb{E}\hat{\nu}_n^t(v) = tv^{\alpha-1}\mathcal{K}'\Gamma(1-\alpha)$ for $d=2$. Together with (3.5.23), this finishes the proof of the convergence of $\mathbb{E}\hat{\nu}_n^t(v)$.

Concentration of $\hat{\nu}_n^t(v)$

We show that, $\mathbb{P}$-a.s., $|\hat{\nu}_n^t(u,\infty) - \mathbb{E}\hat{\nu}_n^t(u,\infty)|$ tends to zero as $N\to\infty$. By a second order Chebyshev inequality it suffices prove that the variance of $\hat{\nu}_n^t(v)$ vanishes fast enough such that it is summable in N. The variance of $\hat{\nu}_n^t(v)$ is given by

$$\left[\mathbb{E}\left(1-\mathcal{E}e^{-vZ_n}\right)^2 - \left(\mathbb{E}\left(1-\mathcal{E}e^{-vZ_n}\right)\right)^2\right]\sum_{x\in B_{d_n(t)}}(k_n(t)\pi_n^t(x))^2 \tag{3.5.31}$$

$$+\ \sum_{x\neq x'}\left[\mathbb{E}\left(\mathcal{E}_x e^{-vZ_n}\mathcal{E}_{x'}e^{-vZ_n}\right) - \left(\mathbb{E}\mathcal{E}e^{-vZ_n}\right)^2\right]k_n(t)\pi_n^t(x)k_n(t)\pi_n^t(x'). \tag{3.5.32}$$

We first construct a bound for (3.5.31). The first term in (3.5.31) is bounded by

$$\mathbb{E}(1-\mathcal{E}e^{-vZ_n})^2 = 2(1-\mathbb{E}\mathcal{E}e^{-vZ_n}) - (1-\mathbb{E}(\mathcal{E}e^{-vZ_n})^2). \tag{3.5.33}$$

Let J' be an independent copy of J. Then the second term can be written as

$$1-\mathbb{E}\left(\mathcal{E}e^{-vZ_n}\right)^2 = 1-\mathbb{E}\mathcal{E}e^{-vZ_n}\mathcal{E}e^{-vZ_n'}. \tag{3.5.34}$$

We proceed as in Section 3.5.1 and count how often the two independent chains J and J' revisit points. Let $\ell_{\theta_n}(J,x) = \#\{0\leq i\leq\theta_n :\ J(i)=x\}$ be the local time of J in x during the first θ_n steps and use the same notation for J'. Write $k_s = \#\{x\in\mathbb{Z}^d : \ell_{\theta_n}(J,x)=s, \ell_{\theta_n}(J',x)>0\}$ and $k_s' = \#\{x\in\mathbb{Z}^d : \ell_{\theta_n}(J',x)=s, \ell_{\theta_n}(J,x)>0\}$. Let $m_s(J,J') \equiv n_s\left(\{J(i)\}_{i=0}^{\theta_n}\cap\{J'(i)\}_{i=0}^{\theta_n}\right)$ be the number of points that have been visited s times by J and J' together. As in Section 3.5.1, we have that

$$
\begin{aligned}
(3.5.34) &= \sum_{\substack{\{n_s,n_s'\}\\ \{m_s\}}} PP'\left(\{n_s\},\{n_s'\},\{m_s\}\right)\\
&\quad \cdot\left(1-\prod_{s=1}^{\theta_n}\left(\mathbb{E}\left(\tfrac{1}{1+\tau(0)v/c_n}\right)^s\right)^{n_s+n_s'+m_s-k_s-k_s'}\right)\\
&= \sum_{\substack{\{n_s,n_s'\}\\ \{m_s\}}} c_n^{-\alpha} PP'\left(\{n_s\},\{n_s'\},\{m_s\}\right)\\
&\quad \cdot\sum_{s=1}^{\theta_n}\left(n_s+n_s'+m_s-k_s-k_s'\right)v^\alpha \tfrac{\Gamma(s+\alpha)\Gamma(1-\alpha)}{\Gamma(s)}+O\left(\theta_n a_n^{-2}\right)\\
&\geq (v/c_n)^\alpha\left(\sum_{\{n_s,n_s'\}} PP'\left(\{n_s\},\{n_s'\}\right)\sum_{s=1}^{\theta_n}(n_s+n_s')\tfrac{\Gamma(s+\alpha)\Gamma(1-\alpha)}{\Gamma(s)}\right. &(3.5.35)\\
&\quad \left.-\sum_{\substack{\{n_s,n_s'\}\\ \{m_s\}}} PP'\left(\{n_s\},\{n_s'\},\{m_s\}\right)\sum_{s=1}^{\theta_n}\left(k_s+k_s'\right)cs^\alpha\right), &(3.5.36)
\end{aligned}
$$

where $c\in(0,\infty)$ is such that $\Gamma(s+\alpha)\leq cs^\alpha\Gamma(s)$ for all $s\geq 1$. Since J' is an independent copy of J, we have that

$$
(3.5.35) = 2v^\alpha c_n^{-\alpha}\sum_{\{n_s\}} P\left(\{n_s\}\right)\sum_{s=1}^{\theta_n} n_s\tfrac{\Gamma(s+\alpha)\Gamma(1-\alpha)}{\Gamma(s)} = 2\left(1-\mathbb{E}\mathcal{E}e^{-vZ_n}\right). \qquad (3.5.37)
$$

Thus,

$$
(3.5.33) \leq v^\alpha c_n^{-\alpha}\sum_{\substack{\{n_s,n_s'\}\\ \{m_s\}}} PP'\left(\{n_s\},\{n_s'\},\{m_s\}\right)\sum_{s=1}^{\theta_n}\left(k_s+k_s'\right)cs^\alpha. \qquad (3.5.38)
$$

Let $s_{max}=C\log\theta_n$, where $C\in(0,\infty)$ will be chosen later. We divide the sum over s in (3.5.38) into two parts. The first is the sum over $s\leq s_{max}$. We bound $s^\alpha\leq s_{max}$ and note that $\sum_{s=1}^{s_{max}}(k_s+k_s')\leq\mathcal{I}_{\theta_n}$, where $\mathcal{I}_{\theta_n}$ is the intersection range of J,J'. Thus, the sum over $s\leq s_{max}$ in (3.5.38) is, up to constants, bounded above by

$$
\begin{aligned}
&v^\alpha c_n^{-\alpha}s_{max}^\alpha\sum_{\{n_s,n_s',m_s\}} PP'\left(\{n_s\},\{n_s'\},\{m_s\}\right)\sum_{s=1}^{s_{max}}\left(k_s+k_s'\right)\\
\leq\ &v^\alpha c_n^{-\alpha}s_{\max}^\alpha EE'\mathcal{I}_{\theta_n}. &(3.5.39)
\end{aligned}
$$

We construct now a bound for $EE'\mathcal{I}_{\theta_n}$. Let δ be the parameter such that the moment generating function of π_Y exists for $|t|\leq\delta$. We write

$$
\begin{aligned}
EE'\mathcal{I}_{\theta_n} &= \sum_{x\in\mathbb{Z}^d}(P(\ell_{\theta_n}(x)>0))^2\\
&\leq \sum_{|x|\leq\delta\theta_n/4}(P(\ell_{\theta_n}(x)>0))^2+\sum_{|x|>\delta\theta_n/4}P(\ell_{\theta_n}(x)>0). &(3.5.40)
\end{aligned}
$$

Since $P(\ell_{\theta_n(x)}>0)=P(\ell_{\theta_n(x)}\geq 1)$, the second term in (3.5.40) is by a first order Chebyshev inequality smaller than

$$
\begin{aligned}
\sum_{|x|>\delta\theta_n/4}E\ell_{\theta_n}(x) &= \sum_{k=1}^{\theta_n}\sum_{|x|>\delta\theta_n/4}P(J(k)=x)=\sum_{k=1}^{\theta_n}P(J(k)\notin B_{\delta\theta_n/4})\\
&\leq \theta_n P(J(k)\notin B_{\delta\theta_n/4})\leq c_2\theta_n e^{-c_1\theta_n}, &(3.5.41)
\end{aligned}
$$

where we used (3.5.8) in the last step. Note that this tends to zero. Let us bound the first summand in (3.5.40). By (3.5.14) one can show that there exists $C\in(0,\infty)$ such that, for all increasing sequences m_n, we have for $0<|x|\leq\delta m_n^{1/2}$

$$
E\ell_{m_n}(x)\leq C\big(\log(m_n/|x|^2)\mathbb{1}_{d=2}+|x|^{d-2}\mathbb{1}_{d\geq 3}\big). \qquad (3.5.42)
$$

Moreover, for $\delta m_n^{1/2}<|x|\leq\delta m_n/4$, we get by (3.5.8)

$$
E\ell_{m_n}(x)\leq Ce^{-c_1|x|^2/m_n}. \qquad (3.5.43)
$$

As in the proof of Lemma 3.19 one can show by (3.5.42) and (3.5.43) that

$$P(\ell_{\theta_n}(x) > 0) \leq \begin{cases} (1 - \frac{\log|x|^2}{\log \theta_n})\mathbb{1}_{d=2} + |x|^{d-2}\mathbb{1}_{d\geq 3}, & 0 < |x| \leq \delta\theta_n^{1/2} \\ e^{-c_1|x|^2/\theta_n}, & \delta\theta_n^{1/2} < |x| \leq \delta\theta_n/4. \end{cases} \tag{3.5.44}$$

Moreover, by (3.5.44) there exists $C'' \in (0, \infty)$ such that the first summand in (3.5.40) is bounded above by

$$C''(\theta_n(\log\theta_n)^{-2}\mathbb{1}_{d=2} + \theta_n^{1/2}\mathbb{1}_{d\geq 3}). \tag{3.5.45}$$

Using (3.5.45) and (3.5.41) in (3.5.39),

$$(3.5.39) \leq tv^\alpha C''(c_n^{-\alpha}\theta_n^{1/2}\log\theta_n\mathbb{1}_{d\geq 3} + c_n^{-\alpha}\theta_n(\log\theta_n)^{-2+\alpha}\mathbb{1}_{d=2}) \equiv \rho_n(d). \tag{3.5.46}$$

It remains to bound the sum over $s > s_{max}$ in (3.5.38). Since the two copies are independent we have that $PP'(\{n_s\}, \{n'_s\}, \{m_s\})(k_s + k'_s) \leq 2n_sP(n_s)$ and hence the sum over $s > s_{max}$ in (3.5.38) is smaller than

$$2v^\alpha c_n^{-\alpha}\sum_{\{n_s\}} P(\{n_s\})\sum_{s=s_{max}}^{\theta_n} n_s s^\alpha = 2v^\alpha c_n^{-\alpha}\sum_{s=s_{max}}^{\theta_n} s^\alpha E n_s(J). \tag{3.5.47}$$

From the proof of Proposition 2.1 in [7], more precisely (2.4), we can deduce for all s that

$$\lim_{n\to\infty}\theta_n^{-1}\gamma_n^{-2}(1-\gamma_n)^{1-s}En_s(J) = 1, \tag{3.5.48}$$

where $\gamma_n = P(J(k) \neq 0, k = 1, \ldots, n)$, and so

$$(3.5.47) \leq 2v^\alpha c_n^{-\alpha}\theta_n^\alpha\sum_{s=s_{max}}^{\theta_n}\gamma_n^2(1-\gamma_n)^{s-1} = 2v^\alpha c_n^{-\alpha}\theta_n^\alpha(1-\gamma_n)^{s_{max}-1}. \tag{3.5.49}$$

We distinguish between $d = 2$ and $d \geq 3$. In the first case we know by [30] that $\gamma_n = O(1/\log\theta_n)$, and therefore, since $s_{max} = C\log\theta_n$, we can choose $C > 0$ such that (3.5.49) $\leq C\theta_n^\alpha$. When $d \geq 3$ we know by transience that $\lim_{n\to\infty}\gamma_n = \gamma \in (0, 1)$ and therefore we can find $C > 0$ so that (3.5.49) $\leq \theta_n^{-2}$. Together with (3.5.46) we get that

$$\mathbb{E}\left(1 - \mathcal{E}e^{-vZ_n}\right)^2 - \left(\mathbb{E}\left(1 - \mathcal{E}e^{-vZ_n}\right)\right)^2 \leq \rho_n(d). \tag{3.5.50}$$

Inserting this into (3.5.31) yields

$$(3.5.31) \leq \rho_n(d)\sum_{x\in B_{d_n(t)}}(k_n(t)\pi_n^t(x))^2. \tag{3.5.51}$$

We construct now a bound for $k_n(t)\pi_n^t(x)$. Let $x \in B_{d_n(t)} \setminus \{0\}$; the claim for $x = 0$ can be deduced from x such that $|x| = 1$. By the definition of $d_n(t)$ and (3.5.14) we know that

$$\begin{aligned} k_n(t)\pi_n^t(x) &= \sum_{k=1}^{k_n(t)-1} P(J(k\theta_n) = x) \leq c_3\int_1^{k_n(t)}(\theta_n z)^{-d/2}e^{-c_4|x|^2/(2\theta_n z)}dz \\ &\leq \theta_n^{-1}E\ell_{\lfloor a_n t\rfloor}(x). \end{aligned} \tag{3.5.52}$$

We use (3.5.42) and (3.5.43) to bound $E\ell_{\lfloor a_n t\rfloor}(x)$ and find that

$$\begin{aligned} (3.5.51) &\leq \rho_n(d)\sum_{x\in B_{d_n(t)}}\theta_n^{-2}(E\ell_{\lfloor a_n t\rfloor}(x))^2 \\ &\leq \rho_n(d)\theta_n^{-2}(d_n(t))^2((\log a_n)^2\mathbb{1}_{d=2} + \mathbb{1}_{d\geq 3}), \end{aligned} \tag{3.5.53}$$

which tends to zero. By construction of θ_n this is summable in N.

Let us construct an upper bound for (3.5.32). More precisely, we show that

$$(3.5.32) \leq \bar{\rho}_n(d) \equiv C''tv^\alpha(\log\theta_n^{-1}\log\log\theta_n\mathbb{1}_{d=2} + \theta_n^{-1/2}\log\theta_n\mathbb{1}_{d\geq 3}). \tag{3.5.54}$$

We distinguish several cases with respect to $|x-x'|$. Let first $|x-x'| > 2K(\theta_n \log\theta_n)^{1/2}$ where $K>0$ will be chosen later. As in the construction of the bound for the variance of $1-\mathcal{E}e^{-vZ_n}$,

$$\begin{aligned} &\mathbb{E}(\mathcal{E}_x e^{-vZ_n}\mathcal{E}_{x'}e^{-vZ_n}) - (\mathbb{E}\mathcal{E}e^{-vZ_n})^2 \\ \leq\ & cv^\alpha c_n^{-\alpha} \textstyle\sum_{\{n_s,n_s',m_s\}^*} PP_{x-x'}\left(\{n_s\},\{n_s'\},\{m_s\}\right)\sum_{s=1}^{\theta_n} s^\alpha(k_s+k_s') \\ \leq\ & cv^\alpha c_n^{-\alpha}\theta_n^\alpha EE_{x-x'}\mathcal{I}_{\theta_n}, \end{aligned} \tag{3.5.55}$$

where $EE_{x-x'}\mathcal{I}_{\theta_n}$ is the expected number of points that are visited by J and an independent copy J' when $J(0)=0$ and $J'(0)=x-x'$. Now,

$$\begin{aligned} EE_{x-x'}\mathcal{I}_{\theta_n} &\leq \theta_n PP_{x-x'}(\{J(i)\}_{i=1}^{\theta_n}\cap\{J'(i)\}_{i=1}^{\theta_n}\neq\emptyset) \\ &\leq c_2\Big(\theta_n^{-K/2c_1+1}\mathbb{1}_{|x-x'|\leq 2\theta_n^{3/4}} + e^{-c_1/6\sqrt{\theta_n}}\mathbb{1}_{|x-x'|>2\theta_n^{3/4}}\Big), \end{aligned} \tag{3.5.56}$$

where we bounded the probability that J and J' visit the same points by the probability that either one of them goes further than $K(\theta_n\log\theta_n)^{1/2}$, respectively $2\theta_n^{3/4}$ from its starting point using (3.5.8). Therefore, for $|x-x'|>2K(\theta_n\log\theta_n)^{1/2}$

$$\begin{aligned} &\mathbb{E}(\mathcal{E}_x e^{-vZ_n}\mathcal{E}_{x'}e^{-vZ_n}) - (\mathbb{E}\mathcal{E}e^{-vZ_n})^2 \\ \leq\ & cc_2 v^\alpha c_n^{-\alpha}\Big(\theta_n^{-c_1K+1+\alpha}\mathbb{1}_{|x-x'|\leq 2\theta_n^{3/4}} + e^{-c_1\sqrt{\theta_n}}\mathbb{1}_{|x-x'|>2\theta_n^{3/4}}\Big). \end{aligned} \tag{3.5.57}$$

We deduce from (3.5.57) that the sum over all x,x' such that $|x-x'|>2\theta_n^{3/4}$ in (3.5.32) is bounded above by $a_n t e^{-c_1/8\sqrt{\theta_n}}$, as claimed in (3.5.54). For x,x' such that $2K(\theta_n\log\theta_n)^{1/2} < |x-x'|2 \leq \theta_n^{3/4}$ we use (3.5.52) to find for $c'\in(0,\infty)$ such that

$$\begin{aligned} &\textstyle\sum_{|x-x'|\leq\theta_n^{3/4}}(k_n(t))^2\pi_n^t(x)\pi_n^t(x')\left[\mathbb{E}(\mathcal{E}_x e^{-vZ_n}\mathcal{E}_{x'}e^{-vZ_n}) - (\mathbb{E}\mathcal{E}e^{-vZ_n})^2\right] \\ \leq\ & c't\theta_n^{-c_1K+\alpha}v^\alpha \textstyle\sum_{x\in B_{d_n(t)}}\pi_n^t(x)\sum_{|x'|\leq\theta_n^{3/4}}k_n(t)\pi_n^t(x') \\ \leq\ & c'tv^\alpha\theta_n^{1/2+\alpha-c_1K}\log a_n. \end{aligned} \tag{3.5.58}$$

For $K>4/c_1$ this is smaller than θ_n^{-2}, which is as desired in (3.5.54). We construct now a bound for the sum over x,x' such that $|x-x'|\leq 2K\sqrt{\theta_n\log\theta_n}$ in (3.5.32). Let first $d\geq 3$. By Cauchy Schwarz inequality and (3.5.52) we have

$$\begin{aligned} &\textstyle\sum_{|x-x'|\leq K\sqrt{\theta_n\log\theta_n}}(k_n(t))^2\pi_n^t(x)\pi_n^t(x')\left[\mathbb{E}(\mathcal{E}_x e^{-vZ_n}\mathcal{E}_{x'}e^{-vZ_n}) - (\mathbb{E}\mathcal{E}e^{-vZ_n})^2\right] \\ \leq\ & v^\alpha\rho_n(d)k_n(t)\textstyle\sum_{|x|\leq K\sqrt{\theta_n\log\theta_n}}k_n(t)\pi_n^t(x) \\ \leq\ & v^\alpha k_n(t)\rho_n(d)K^2\log\theta_n, \end{aligned} \tag{3.5.59}$$

as desired in (3.5.54). Let $d=2$. We distinguish one additional case w.r.t. the size of $|x-x'|$. Let first x,x' be such that $k\sqrt{\theta_n\log\log\theta_n}\leq|x-x'|\leq K\sqrt{\theta_n\log\theta_n}$, where $k>0$ will be chosen later. As in the bound for the variance of $1-\mathcal{E}e^{-vZ_n}$ we get

$$\begin{aligned} &\mathbb{E}(\mathcal{E}_x e^{-vZ_n}\mathcal{E}_{x'}e^{-vZ_n}) - (\mathbb{E}\mathcal{E}e^{-vZ_n})^2 \\ \leq\ & v^\alpha c_n^{-\alpha}(s_{max}^\alpha(2)EE_{x-x'}\mathcal{J}_n + C\theta_n^\alpha\mathbb{1}_{d=2}+\theta_n^{-2}\mathbb{1}_{d\geq 3}) \\ \leq\ & 2v^\alpha c_n^{-\alpha}\theta_n(\log\theta_n)^{-c_1k/2+2\alpha}, \end{aligned} \tag{3.5.60}$$

where the second step follows as in (3.5.57). Let $k>6/c_1$ now. Together with (3.5.52),

$$\begin{aligned} &\textstyle\sum_{|x-x'|\leq K\sqrt{\theta_n\log\theta_n}}(k_n(t))^2\pi_n^t(x)\pi_n^t(x')\left[\mathbb{E}(\mathcal{E}_x e^{-vZ_n}\mathcal{E}_{x'}e^{-vZ_n}) - (\mathbb{E}\mathcal{E}e^{-vZ_n})^2\right] \\ \leq\ & k_n(t)2v^\alpha c_n^{-\alpha}\theta_n(\log\theta_n)^{-c_1k/2+2\alpha}\textstyle\sum_{|x'|\leq K\sqrt{\theta_n\log\theta_n}}k_n(t)\pi_n^t(x') \\ \leq\ & k_n(t)2v^\alpha c_n^{-\alpha}\theta_n(\log\theta_n)^{-c_1k/2+2\alpha+1}K^2, \end{aligned} \tag{3.5.61}$$

as claimed in (3.5.54). Finally, by Cauchy Schwarz inequality, the sum over x, x' such that $|x - x'| \leq k\sqrt{\theta_n \log\log\theta_n}$ is bounded by

$$\begin{aligned}&\textstyle\sum_{|x-x'|\leq k\sqrt{\theta_n \log\log\theta_n}}(k_n(t))^2\pi_n^t(x)\pi_n^t(x')\left[\mathbb{E}(\mathcal{E}_x e^{-vZ_n}\mathcal{E}_{x'}e^{-vZ_n}) - (\mathbb{E}\mathcal{E}e^{-vZ_n})^2\right]\\ &\leq v^\alpha k_n(t)\rho_n(d)k^2\log\log\theta_n,\end{aligned} \tag{3.5.62}$$

as desired in (3.5.54). This finishes the proof of (3.5.54) and so $\mathbb{E}(\hat{\nu}_n^t(v))^2 - (\mathbb{E}\hat{\nu}_n^t(v))^2 \leq \bar{\rho}_n(d)$. By a second order Chebyshev inequality and Borel-Cantelli Lemma this proves that, $\mathbb{P}$-a.s., $|\hat{\nu}_n^t(v) - \mathbb{E}\hat{\nu}_n^t(v)|$ vanishes. Together with Section 3.5.1 we conclude that, $\mathbb{P}$-a.s., the Laplace transform of $\nu_n^t(u)$ converges to that of $t\nu(u,\infty)$, proving that, $\mathbb{P}$-a.s.,

$$\lim_{N\to\infty}\nu_n^t(u) = t\nu(u,\infty). \tag{3.5.63}$$

The verification of (D-2) is complete.

3.5.2 Verification of (D-3)

In this section we show that, $\mathbb{P}$-a.s., (D-3) is satisfied. As in Section 3.5.1 we consider the Laplace transform for σ_n^t. For $v, v' > 0$ it is given by

$$\hat{\sigma}_n^t(v,v') \equiv (vv')^{-1}\textstyle\sum_{x\in B_{d_n(t)}} k_n(t)\pi_n^t(x)\left(1 - \mathcal{E}_x e^{-vZ_n}\right)\left(1 - \mathcal{E}_x e^{-v'Z_n}\right). \tag{3.5.64}$$

Taking expectation with respect to the random environment we find by (3.5.50) that

$$\begin{aligned}\mathbb{E}\hat{\sigma}_n^t(v,v') &= (vv')^{-1}\textstyle\sum_{x\in B_{d_n(t)}} k_n(t)\pi_n^t(x)\mathbb{E}\left(\left(1-\mathcal{E}e^{-vZ_n}\right)\left(1-\mathcal{E}e^{-v'Z_n}\right)\right)\\ &\leq (vv')^{-1}\mathbb{E}\left(1-\mathcal{E}e^{-\max\{v,v'\}Z_n}\right)^2\textstyle\sum_{x\in B_{d_n(t)}} k_n(t)\pi_n^t(x)\\ &\leq (vv')^{-1}\rho_n(d)k_n(t),\end{aligned} \tag{3.5.65}$$

which by the definition of $\rho_n(d)$ is smaller than $\bar{\rho}_n(d)$ and tends to zero such that it is summable in N. By a first order Chebyshev inequality and Borel Cantelli Lemma this shows that, $\mathbb{P}$-a.s., (D-3) is satisfied.

3.5.3 Verification of (D-4)

Let us now establish that, $\mathbb{P}$-a.s., (D-4) is fulfilled. Fix $\varepsilon > 0$. Notice that

$$\begin{aligned}\overline{m}_n^t(\varepsilon) &= c_n^{-1}k_n(t)\textstyle\sum_{x\in B_{d_n(t)}}\pi_n^t(x)\mathcal{E}_x Z_n \mathbb{1}_{Z_n\leq c_n\varepsilon}\\ &\leq c_n^{-1}k_n(t)\textstyle\sum_{x\in B_{d_n(t)}}\pi_n^t(x)\sum_{y\in\mathbb{Z}^d}\mathcal{E}_x\sum_{j=1}^{\ell_{\theta_n}(y)} e_j\tau(y)\mathbb{1}_{\sum_{j=1}^{\ell_{\theta_n}(y)} e_j\tau(y)\leq c_n\varepsilon}.\end{aligned} \tag{3.5.66}$$

We show now that the right hand side of (3.5.66) is, $\mathbb{P}$-a.s., bounded above by $K't\varepsilon^{1-\alpha}$. Since (3.5.66) is true for all $\omega\in\Omega$ this implies that (D-4) holds true. Fix $x \in B_{d_n(t)}$ and $y \in \mathbb{Z}^d$. We take expectation w.r.t. the random environment, apply Fubini, and obtain that the expectation of each summand in the right hand side of (3.5.66) is bounded by

$$\begin{aligned}c_n^{-1}\mathbb{E}\mathcal{E}\left[\textstyle\sum_{j=1}^{\ell_{\theta_n}(y-x)} e_j\tau(y)\mathbb{1}_{\sum_{j=1}^{\ell_{\theta_n}(y-x)} e_j\tau(y)\leq c_n\varepsilon}\right] &= \varepsilon\,\mathcal{E}\left[\int_0^1 dz\mathbb{P}\left(\tau(0) > \frac{c_n\varepsilon z}{\sum_{j=1}^{\ell_{\theta_n}(y-x)} e_j}\right)\right]\\ &\leq c'c_n^{-\alpha}\varepsilon^{1-\alpha}\mathcal{E}\left(\textstyle\sum_{j=1}^{\ell_{\theta_n}(y-x)} e_j\right)^\alpha\\ &\leq c'c_n^{-\alpha}\varepsilon^{1-\alpha}E\left(\frac{\Gamma(\ell_{\theta_n}(x-y)+\alpha)}{\Gamma(\ell_{\theta_n}(x-y))}\right),\end{aligned} \tag{3.5.67}$$

where $c' \in (0,\infty)$ and where we used the fact that $\sum_{j=1}^{k} e_j$ has gamma distribution with parameters $(k,1)$. We take the sum over all $y \in \mathbb{Z}^d$ to find

$$\begin{aligned} c' c_n^{-\alpha} \sum_{y\in\mathbb{Z}^d} \varepsilon^{1-\alpha} E\left(\tfrac{\Gamma(\ell_{\theta_n}(x-y)+\alpha)}{\Gamma(\ell_{\theta_n}(x-y))}\right) &= c' c_n^{-\alpha}\varepsilon^{1-\alpha} \sum_{y\in\mathbb{Z}^d} E\left(\tfrac{\Gamma(\ell_{\theta_n}(y)+\alpha)}{\Gamma(\ell_{\theta_n}(y))}\right) \\ &\le C' \sum_{y\in\mathbb{Z}^d} E(\ell_{\theta_n}(y))^\alpha, \end{aligned} \tag{3.5.68}$$

where $C' \in (0,\infty)$. By Theorem 1 in [30], respectively Proposition 2.1 in [7], this is bounded above by to $C''\theta_n((\log\theta_n)^{\alpha-1}\mathbb{1}_{d=2} + \mathbb{1}_{d\ge 3})$ for $C'' \in (0,\infty)$. Therefore, for $K' = C'C''$,

$$c_n^{-1} k_n(t) \sum_{x\in B_{d_n(t)}} \pi_n^t(x) \sum_y \mathbb{E}\mathcal{E}_x\Big[\sum_{j=1}^{\ell_{\theta_n}(y)} e_j\tau(y)\mathbb{1}_{\sum_{j=1}^{\ell_{\theta_n}(y)} e_j\tau(y)\le c_n\varepsilon}\Big] \le K' t\varepsilon^{1-\alpha}. \tag{3.5.69}$$

Condition (D-4) follows if we can show that the right hand side of (3.5.66) concentrates, $\mathbb{P}$-a.s., around its mean value. To this end we show that the variance of the right hand side of (3.5.66) can be bounded as the variance of $\hat{\nu}_n^t(v)$. Let us first construct a bound for the variance of each summand in (3.5.66). Namely,

$$\mathbb{E}c_n^{-1}\Big(\sum_y \mathcal{E}\Big[\sum_{j=1}^{\ell_{\theta_n}(y)} e_j\tau(y)\mathbb{1}_{\sum_{j=1}^{\ell_{\theta_n}(y)} e_j\tau(y)\le c_n\varepsilon}\Big]\Big)^2$$

$$\le \sum_y \mathbb{E}\Big(c_n^{-1}\mathcal{E}\Big[\sum_{j=1}^{\ell_{\theta_n}(y)} e_j\tau(y)\mathbb{1}_{\sum_{j=1}^{\ell_{\theta_n}(y)} e_j\tau(y)\le c_n\varepsilon}\Big]\Big)^2 \tag{3.5.70}$$

$$+\ c_n^{-2}\sum_{y,y':y\neq y'} E\ell_{\theta_n}(y)E\ell_{\theta_n}(y')\Big(\mathbb{E}\mathcal{E}e_1\tau(0)\mathbb{1}_{e_1\tau(0)\le c_n\varepsilon}\Big)^2, \tag{3.5.71}$$

where (3.5.71) follows from

$$\mathcal{E}\Big[\sum_{j=1}^{\ell_{\theta_n}(y)} e_j\tau(y)\mathbb{1}_{\sum_{j=1}^{\ell_{\theta_n}(y)} e_j\tau(y)\le c_n\varepsilon}\Big] \le E\ell_{\theta_n}(y)\mathcal{E}\Big[e_1\tau(y)\mathbb{1}_{e_1\tau(y)\le c_n\varepsilon}\Big]. \tag{3.5.72}$$

We show now that both, (3.5.70) and (3.5.71) are bounded above by $\rho_n(d)$. Let us begin with a bound for (3.5.70). By construction of the indicator function we have for $y \in \mathbb{Z}^d$

$$c_n^{-1}\mathcal{E}\Big[\sum_{j=1}^{\ell_{\theta_n}(y)} e_j\tau(y)\mathbb{1}_{\sum_{j=1}^{\ell_{\theta_n}(y)} e_j\tau(y)\le c_n\varepsilon}\Big] \le \varepsilon E\mathbb{1}_{\ell_{\theta_n}(y)>0} = P(\ell_{\theta_n}(y)>0). \tag{3.5.73}$$

Now, by (3.5.73) and (3.5.68) we get for $c = c'\varepsilon^{1-\alpha}$

$$\begin{aligned} (3.5.70) &\le c'\varepsilon^{1-\alpha}c_n^{-\alpha}\sum_{y\in\mathbb{Z}^d} P(\ell_{\theta_n}(y)>0)E\left(\tfrac{\Gamma(\ell_{\theta_n}(x-y)+\alpha)}{\Gamma(\ell_{\theta_n}(x-y))}\right) \\ &\le \sum_{y\in\mathbb{Z}^d} cc_n^{-\alpha}P(\ell_{\theta_n}(y)>0)E\left(\ell_{\theta_n}(y)\right)^\alpha \\ &\le cc_n^{-\alpha}\big(s_{max}^\alpha\sum_{y\in\mathbb{Z}^d}(P(\ell_{\theta_n}(y)>0))^2 + \sum_{y\in\mathbb{Z}^d} E\left[\left(\ell_{\theta_n}(y)\right)^\alpha\mathbb{1}_{\ell_{\theta_n}(y)>s_{max}}\right]\big) \\ &= cc_n^{-\alpha}\big(s_{max}^\alpha EE'\mathcal{I}_{\theta_n} + \textstyle\sum_{s=s_{max}}^{\theta_n} s^\alpha En_s(J)\big). \end{aligned} \tag{3.5.74}$$

which by (3.5.39) and the calculations after (3.5.47) is bounded above by $\rho_n(d)$. Let us now establish that (3.5.71) is of smaller order than (3.5.70). We know that $\sum_{y\neq y'} E\ell_{\theta_n}(y)E\ell_{\theta_n}(y') \le \left(\sum_y E\ell_{\theta_n}(y)\right)^2$, which by Theorem 1 in [30], respectively Proposition 2.1 in [7], is bounded by θ_n^2 and so

$$(3.5.71) \le 2c_n^{-2\alpha}(\theta_n)^2. \tag{3.5.75}$$

Together with (3.5.74) this proves that

$$\mathbb{E}\left(c_n^{-1}\mathcal{E}Z_n\mathbb{1}_{Z_n\le c_n\varepsilon}\right)^2 \le \rho_n(d), \tag{3.5.76}$$

which is the same bound as the one for the variance from Section 3.5.1 (cf. (3.5.39)). Therefore one can proceed as in Section (3.5.1) to prove that the variance of (3.5.66) is bounded above by $\bar{\rho}_n(d)$. Hence, the right hand side of (3.5.66) concentrates around its mean and the verification of (D-4) is finished.

3.5.4 Conclusion of the proof of Theorem 3.7

In Sections 3.5.1-3.5.3 we verified that (D-2)-(D-4) are $\mathbb{P}$-a.s. satisfied for $u > 0$, $t > 0$, and $\varepsilon > 0$. Together with Proposition 3.20, which shows that (D-2)-(D-4)⇒ (A-2)-(A-4) and that (A-0)-(A-1) hold, we know that we may apply Theorem 3.1 and obtain that, $\mathbb{P}$-a.s., $S_n^{J,b} \stackrel{J_1}{\Longrightarrow} V_\nu$ where V_ν is a subordinator with Lévy measure ν. This finishes the proof of Theorem 3.7.

3.6 Aging in B(A)TM

In this section we present the proofs of Theorem 3.5, Theorem 3.6, and Theorem 3.8. Section 3.6.1, respectively Section 3.6.2 and Section 3.6.3, contains the proof of Theorem 3.5 for $i = 1$, respectively $i = 2$ and $i = 3$. We then prove Theorem 3.1.6 in Section 3.6.4. Finally, the proof of Theorem 3.8 is contained in Section 3.6.5.

The proofs in Sections 3.6.1-3.6.3 follow a common scheme. We show that for each $i \in \{1, 2, 3\}$, as $s \to \infty$, $\mathcal{C}_s^i(1, \rho)$, coincides with the probability of the event $\mathcal{A}_{s,\rho} \equiv \{\mathcal{R}_s \cap (1, 1+\rho) = \emptyset\}$, where $\mathcal{R}_s = \{S_s^{\widetilde{J},b}(t), t \geq 0\}$ is the range of $S_s^{\widetilde{J},b}$. We then use that $\mathbb{P}$-a.s.,

$$\lim_{s\to\infty} \mathcal{P}(\mathcal{A}_{s,\rho}) = \mathrm{Asl}_\alpha(1/(1 + \rho)). \tag{3.6.1}$$

The proof of (3.6.1) closely follows that of Theorem 1.6 in [37]. We thus only sketch it here. Namely, it relies on the continuity of the overshoot function that maps $Y \in D[0, \infty)$ to $\chi_u(Y) = Y(\mathcal{L}_u(Y)) - u$, where $\mathcal{L}_u$ is the first passage to the level $u > 0$ of Y. For Lévy motions having $\mathcal{P}$-a.s. diverging paths, this mapping is $\mathcal{P}$-a.s. continuous on $D[0, \infty)$ equipped with Skorohod's J_1 topology. Now, $\mathcal{A}_{s,\rho} = \{\chi_1(S_s^{\widetilde{J},b}) \geq 1 + \rho\}$ and by Theorem 3.4, $\mathbb{P}$-a.s., $S_s^{\widetilde{J},b} \stackrel{J_1}{\Longrightarrow} V_\alpha$. Since V_α has $\mathcal{P}$-a.s. diverging paths we deduce that, $\mathbb{P}$-a.s.,

$$\lim_{s\to\infty} \mathcal{P}(\mathcal{A}_{s,\rho}) = \mathcal{P}(\xi_1(V_\alpha) \geq 1 + \rho) = \mathrm{Asl}_\alpha(1/(1 + \rho)), \quad \rho > 0, \tag{3.6.2}$$

where the last equality follows from the arcsine law for stable subordinators (see Section III in [21]). Given (3.6.2), it remains to establish that, $\mathbb{P}$-a.s.,

$$\lim_{s\to\infty} |\mathcal{P}(\mathcal{A}_{s,\rho}) - \mathcal{P}(\mathcal{A}_{s,\rho}^i)| = 0, \quad \forall \rho > 0, \tag{3.6.3}$$

where $\mathcal{A}_{s,\rho}^i$ stands for the events appearing in the right hand sides of (3.1.41)-(3.1.43), namely $\mathcal{C}_s^i(1, \rho) = \mathcal{P}(\mathcal{A}_{s,\rho}^i)$. The verification of (3.6.3) is contained in Sections 3.6.1-3.6.3.

3.6.1 Convergence of $\mathcal{C}_s^1(1, \rho)$ in BATM

In this section we prove that (3.6.3) holds for $i = 1$.

Step 1. Let us establish that $\mathcal{P}(\mathcal{A}_{s,\rho}, (\mathcal{A}_{s,\rho}^1)^c)$ vanishes $\mathbb{P}$-a.s. For $k \in \mathbb{N}$ we define

$$A_k \equiv \{\textstyle\sum_{i=1}^k Z_{s,i}^{\widetilde{J}} < 1, \quad \text{and} \quad \sum_{i=1}^{k+1} Z_{s,i}^{\widetilde{J}} > (1 + \rho)\}. \tag{3.6.4}$$

Then, $\mathcal{A}_{s,\rho} = \bigcup_{k\geq 1} A_k$. In fact, for all $\delta > 0$ there exists $M > 0$ such that, $\mathbb{P}$-a.s.,

$$\mathcal{P}(\mathcal{A}_{s,\rho}, (\mathcal{A}_{s,\rho}^1)^c) \leq \mathcal{P}(\textstyle\bigcup_{k\leq k_s(M)} A_k, (\mathcal{A}_{s,\rho}^1)^c) + \delta. \tag{3.6.5}$$

To see the claim of (3.6.5) note that, since $S_s^{\widetilde{J},b} \stackrel{J_1}{\Longrightarrow} V_\alpha$,

$$\mathcal{P}(\textstyle\bigcup_{k\geq k_s(M)} A_k, (\mathcal{A}_{s,\rho}^1)^c) \leq \mathcal{P}(S_s^{\widetilde{J},b}(M) < 1) \leq \mathcal{P}(V_\alpha(M) < 1 + \delta) + \delta, \tag{3.6.6}$$

which vanishes as $M \to \infty$ and proves (3.6.5). Let us now show that on A_k there is $\mathbb{P}$-a.s. only one $x \in T_s$ that contributes to $Z_{s,k+1}$, bound the contribution to $Z_{s,k+1}$ coming from $y \notin T_s$ by $\delta_s = \epsilon_s^{(1-\alpha)/3}$ and show that $\max_{y\sim x} \tau(y) \leq \epsilon_n^{-4}$. Notice that by Lemma 3.14 there exists $x \in T_s$ such that $\widetilde{\ell}_{\theta_s(k+1)}(x) - \widetilde{\ell}_{\theta_s k}(x) > 0$ on A_k because else for all $\delta > 0$, for s large enough, $Z_{s,k+1} \leq \delta$. For $B_k = \{\max_{y\sim x} \tau(y) \mathbb{1}_{x\in T_s, \widetilde{\ell}_{\theta_s(k+1)}(x) - \widetilde{\ell}_{\theta_s k}(x)>0} > \epsilon_s^{-2}\}$,

$$\begin{aligned} &\mathcal{P}\big(\textstyle\bigcup_{k\leq k_s(M)} \{|Z^{\widetilde{J}}_{s,k+1} - \overline{Z}^{\widetilde{J}}_{s,k+1}| > \delta_s, A_k, B_k\}\big) \\ &\leq k_s(M) \textstyle\sum_{z\in B_{d_s(M)}} \pi_s^M(z) \mathcal{P}_z(|Z^{\widetilde{J}}_{s,1} - \overline{Z}^{\widetilde{J}}_{s,1}| > \delta_s, Z^{\widetilde{J}}_{s,1} > \rho, B_0). \end{aligned} \tag{3.6.7}$$

We show now that the right hand side of (3.6.7) vanishes $\mathbb{P}$-a.s. Inserting the $\mathbb{P}$-a.s. bounds for π_s^M of (3.4.70) and (3.4.72) and taking expectation with respect to the random environment in (3.6.7), it remains to bound

$$k_s(M)(\log\log a_s)^3 \mathbb{E}\big[\mathcal{P}(\textstyle\sum_{y\notin T_s} \widetilde{\ell}_{\theta_s}(y)\gamma_s(y) > \delta_s) + \mathbb{1}_{0\in T_s, \max_{y\sim 0}\tau(y) > \epsilon_s^{-4}}\big]. \tag{3.6.8}$$

As in the proof of Lemma 3.14 one can show that the first summand in (3.6.8) is bounded above by δ_s. Moreover, the second summand in (3.6.8) is smaller than $\varepsilon_s^{2\alpha}$. Thus, (3.6.8) vanishes and hence, $\mathbb{P}$-a.s., (3.6.7) $\to 0$. By (3.6.8), $(\mathcal{A}^1_{s,\rho})^c$ can only hold if either $X(s) \notin T_s$ or $X(s(1+\rho)) \notin T_s$ or both are not in T_s. Let us show that the probability of this vanishes as $s \to \infty$. By (3.6.7), we know that $X(v) \in T_s$ for some $s - v \leq \delta_s$ and same is true for $s(1+\rho)$. Thus $\mathcal{P}(\mathcal{A}_{s,\rho} \cap (\mathcal{A}^1_{s,\rho})^c)$ is, up to small enough error, bounded above by

$$\textstyle\sum_{s'=s,s(1+\rho)} \mathcal{P}(X(s') \notin T_s, \exists\, v > 0 \,:\, X(v) \in T_s,\ s' - v \leq \delta_s). \tag{3.6.9}$$

We prove that (3.6.9) tends to zero for $s' = s$, the same proof works for $s' = s(1+\rho)$. Let us distinguish two cases with respect to θ. Let first $\theta > 0$. We establish that, for all $x \in T_s$, when $X(v) = x \in T_s$ then with probability larger than $1-\delta$, $X(v') \in A(x) = \{x\} \cup \{y \sim x\}$ for all $v \leq v' \leq s$. We then use this to conclude that, with probability at least $1-\delta$, $X(s) = x$, proving that (3.6.9) tends to zero. Writing $N_x(A(x))$ for the number of returns to x before $\widetilde{J}$ escapes $A(x)$, we have

$$\begin{aligned} &\mathcal{P}_x(\exists v' \leq \delta_s : X(v') \notin A(x)) \\ &\leq P_x(N_x(A(x)) \leq (\delta_s \theta_s^{1/2})^\theta) + \mathcal{P}_x(\textstyle\sum_{i=1}^{(\delta_s\theta_s^{1/2})^\theta} (\lambda(x))^{-1} e_i \leq \delta_s). \end{aligned} \tag{3.6.10}$$

By (3.6.8), $\max_{y\sim x}(1 - p(y,x)) \leq (s\epsilon_s^3)^{-\theta}$, and so the first probability in (3.6.10) is smaller than $(\delta_s\theta_s^{1/2}/\epsilon_s^3)^\theta < \delta$. Moreover, $(\lambda(x))^{-1} \geq 2ds^{1-\theta}\epsilon_s^{-4\theta}$ and we deduce by the law of large numbers that also the second probability in (3.6.10) vanishes. It remains to bound

$$\mathcal{P}_x(X(v') \in A(x)\ \forall v \leq v' \leq s, X(s) \neq x). \tag{3.6.11}$$

Since $\min_{y\sim x}(\lambda(y))^{-1} \leq \epsilon_s^{-4}(s\epsilon_s)^{-\theta}$, with probability larger than $1 - \exp(-\epsilon_s^{-\theta})$, there exists v' such that $X(v') = x$ and $s - v' \leq s^{-\theta}\epsilon_s^{-4}$. By the Markov property we have for all such v',

$$\mathcal{P}_x(X(s-v') \neq x) \leq \mathcal{P}_x(e_1\lambda^{-1}(x) < s - v') \leq 1 - e^{-s^{-1}\epsilon_s^{-4(1-\theta)}}, \tag{3.6.12}$$

which tends to zero. Thus, (3.6.9) $\to 0$ for $\theta > 0$. When $\theta = 0$, one can bound (3.6.9) directly as in (3.6.12). This shows that $\mathcal{P}(\mathcal{A}_{s,\rho}, (\mathcal{A}^1_{s,\rho})^c) \leq \delta$.

Step 2. Let us now show that, $\mathbb{P}$-a.s., $\mathcal{P}((\mathcal{A}_{s,\rho})^c, \mathcal{A}^1_{s,\rho}) \to 0$. Let $m_{s,\rho} \equiv (S^{\widetilde{J}})^{\leftarrow}(s(1+\rho)) - (S^{\widetilde{J}})^{\leftarrow}(s)$, where $(S^{\widetilde{J}})^{\leftarrow}(t) = \inf\{v \geq 0 : S^{\widetilde{J}}(v) > t\}$. Notice that $(\mathcal{A}_{s,\rho})^c \subseteq \{m_{s,\rho} \geq$

$\theta_s\} \cup \{Z_{s,1} > 1\}$. As in the verification of (A-0) one can show that $\mathcal{P}(Z_{s,1} > \rho)$ tends $\mathbb{P}$-a.s. to zero and so, for all $\delta > 0$ there exists s large enough such that $\mathcal{P}((\mathcal{A}_{s,\rho})^c, \mathcal{A}^1_{s,\rho}) \leq \mathcal{P}(m_{s,\rho} \geq \theta_s, \mathcal{A}^1_{s,\rho}) + \delta$. Let us distinguish whether $d \geq 3$ or $d = 2$. In the first case we use the identity $X(t) = \widetilde{J}((S^{\widetilde{J}})^{\leftarrow}(t))$ and get by (3.3.2) of Theorem 3.10, uniformly in $x \in \mathbb{Z}^d$,

$$\mathcal{P}_x(X(s(1+\rho)) = x, m_{s,\rho} \geq \theta_s) = \mathcal{P}_x(\widetilde{J}(m_{s,\rho}) = x, m_{s,\rho} \geq \theta_s) \leq \textstyle\int_{\theta_s}^{\infty} v^{-d/2} dv, \tag{3.6.13}$$

which is smaller than $\theta_s^{-d/2+1}$ and shows that $\mathcal{P}((\mathcal{A}_{s,\rho})^c, \mathcal{A}^1_{s,\rho}) \to 0$ for $d \geq 3$.

Let $d = 2$. We construct a more precise bound for $\mathcal{P}((\mathcal{A}_{s,\rho})^c \cap \mathcal{A}^1_{s,\rho})$ than $\mathcal{P}(\{m_{s,\rho} \geq \theta_s\} \cap \mathcal{A}^1_{s,\rho}) + \delta$. Assume first that $\mathrm{dist}(\mathcal{R}_s, 1+\rho) > \delta$ and that there are $t, t' > 0$ such that $S_s^{\widetilde{J},b}(t), S_s^{\widetilde{J},b}(t+t') \in (1, 1+\rho-\delta/2)$. Then, $s < S^{\widetilde{J}}(k_s(t)\theta_s) < S^{\widetilde{J}}(k_s(t+t')\theta_s) < s(1+\rho)$ and so $m_{s,\rho} \geq \theta_s(k_s(t+t') - k_s(t'))$. Moreover, by (3.6.6) there exists $M > 0$ such that $m_{s,\rho} \leq \theta_s k_s(M)$. Since $\mathrm{dist}(\mathcal{R}_s, 1+\rho) > \delta$ one can show as in Step 1 that $X(s(1+\rho)) = x \in T_s$. But then, on $(\mathcal{A}_{s,\rho})^c \cap \mathcal{A}^1_{s,\rho}$, we have with probability larger than $1 - (\log\theta_s)^{-2}$ that $\widetilde{\ell}_{m_{s,\rho}}(x) - \widetilde{\ell}_{m_{s,\rho}-\theta_s}(x) > c\log\theta_s/\log\log\theta_s$ for some $c \in (0,\infty)$. $\mathcal{R}_s \cap (1, 1+\rho) = \emptyset$ or $X(s) \neq x$.) By (3.3.2) of Theorem 3.10 and a first order Chebyshev inequality we get, for all $x \in \mathbb{Z}^d$,

$$\begin{aligned}
&\mathcal{P}_x(\widetilde{J}(m_{s,\rho}) = x, m_{s,\rho} \in (\theta_s k_s(t), \theta_s k_s(M)), \widetilde{\ell}_{m_{s,\rho}}(x) - \widetilde{\ell}_{m_{s,\rho}-\theta_s}(x) > \tfrac{c\log\theta_s}{\log\log\theta_s}) \\
&\leq P_x(\widetilde{\ell}_{\theta_s k_s(M)}(x) - \widetilde{\ell}_{\theta_s k_s(t)-\theta_s}(x) > \tfrac{c\log\theta_s}{\log\log\theta_s}) \\
&\leq c\tfrac{c\log\log\theta_s}{\log\theta_s}\log(M/t),
\end{aligned} \tag{3.6.14}$$

which tends, as $s \to \infty$, to zero. It remains to establish that for all $\delta' > 0$ there exist $\delta > 0$, $t > 0$ such that $\mathcal{P}(\mathrm{dist}(\mathcal{R}_s, 1+\rho) > \delta) \leq 1 - \delta'$ and $\mathcal{P}(S_s^{\widetilde{J},b}(t+t') \in (1, 1+\rho-\delta/2) | S_s^{\widetilde{J},b}(t) \in (1, 1+\rho-\delta)) \geq 1 - \delta'$. This can be derived from the convergence of $S_s^{\widetilde{J},b}$ to V_α and properties of V_α (see Section III in [21]). Thus, $\mathcal{P}((\mathcal{A}_{s,\rho})^c, \mathcal{A}^1_{s,\rho}) \leq \delta + 2\delta'$ for $d \geq 2$. Together with Step 1 this finishes the proof of (3.6.3) for $i = 1$.

3.6.2 Convergence of $\mathcal{C}_s^2(1,\rho)$ in BATM

In this section we prove the claim of (3.6.3) for $i = 2$.
Step 1. We show that, $\mathbb{P}$-a.s., $\mathcal{P}(\mathcal{A}_{s,\rho}, (\mathcal{A}^2_{s,\rho})^c) \to 0$. Note that $\mathcal{A}_{s,\rho} \subseteq \{m_{s,\rho} \leq \theta_s\}$. Let $x \in B_{a_s}$. Using $X(s) = \widetilde{J}((S^{\widetilde{J}})^{\leftarrow}(s))$ and the Markov property

$$\mathcal{P}(X(s) = x, (\mathcal{A}^2_{s,\rho})^c, m_{s,\rho} \leq \theta_s) \leq \mathcal{P}(X(s) = x) P_x(\eta(B_{(\theta_s\log\theta_s)^{1/2}}(x)) \leq \theta_s), \tag{3.6.15}$$

where $\eta(B)$ is the exit time of B for $\widetilde{J}$ as defined in Section 3.3.1. By (3.3.8) of Lemma 3.11 we have $\mathbb{P}$-a.s. for all $x \in B_{a_s}$,

$$P_x(\eta(B_{(\theta_s\log\theta_s)^{1/2}}(x)) \leq \theta_s) \leq \exp(-c_4\log\theta_s). \tag{3.6.16}$$

If we can show that $\mathcal{P}(X(s) \notin B_{a_s}) \leq \delta$, then (3.6.15) and (3.6.16) imply that $\mathcal{P}(\mathcal{A}_{s,\rho}(\mathcal{A}^2_{s,\rho})^c) \leq \delta$. To bound $\mathcal{P}(X(s) \notin B_{a_s})$ we recall that by (3.6.6), with probability larger than $1-\delta$, $(S^{\widetilde{J}})^{\leftarrow}(s) \leq k_s(M)\theta_s$ for $M > 0$ and so

$$\mathcal{P}(X(s) \notin B_{a_s}, (S^{\widetilde{J}})^{\leftarrow}(s) \leq a_s M) \leq P(\eta(B_{a_s}) \leq a_s M) \leq e^{-c_4 a_s^{1/2} M^{-2}}, \tag{3.6.17}$$

by (3.3.8) of Lemma 3.11. This tends to zero and we conclude that $\mathcal{P}(\mathcal{A}_{s,\rho}, (\mathcal{A}^2_{s,\rho})^c) \leq \delta$.
Step 2. Now we prove that $\mathcal{P}((\mathcal{A}_{s,\rho})^c, \mathcal{A}^2_{s,\rho})$ vanishes $\mathbb{P}$-a.s. On $(\mathcal{A}_{s,\rho})^c$ one can show as in Section 3.6.1 (Step 2) that there exist t, t' such that $m_{s,\rho} \geq \theta_s(k_s(t+t') - k_s(t)) \gg \theta_s^2$. Moreover, by

(3.6.17) we know that, with probability larger than $1-\delta$, $X(s)\in B_{a_s}$. But by (3.3.9) of Lemma 3.11, $\mathbb{P}$-a.s., for all $x\in B_{a_s}$

$$P_x(\eta(B_{(\theta_s\log\theta_s)^{1/2}}(x))>\theta_s^2)\le\exp(-c_4\log\theta_s). \tag{3.6.18}$$

As in (3.6.15) we thus get, $\mathbb{P}$-a.s., $\mathcal{P}((\mathcal{A}_{s,\rho})^c,\mathcal{A}^2_{s,\rho})\to 0$. Together with Step 1, the proof of (3.6.3) is finished for $i=2$.

3.6.3 Convergence of $\mathcal{C}^3_s(1,\rho)$ in BATM

We now show that (3.6.3) holds for $\mathcal{C}^3_s(1,\rho)$. This follows readily from Sections 3.6.1 and 3.6.2. Indeed on the one hand,

$$\mathcal{P}((\mathcal{A}_{s,\rho})^c,\mathcal{A}^3_{s,\rho})\le\mathcal{P}((\mathcal{A}_{s,\rho})^c,\mathcal{A}^1_{s,\rho})+\mathcal{P}((\mathcal{A}_{s,\rho})^c,\mathcal{A}^2_{s,\rho}), \tag{3.6.19}$$

and on the other hand, $\mathcal{P}(\mathcal{A}_{s,\rho},(\mathcal{A}^3_{s,\rho})^c)\le\mathcal{P}(\mathcal{A}_{s,\rho},(\mathcal{A}^1_{s,\rho})^c)$. Both upper bounds tend by Sections 3.6.1 and 3.6.2 $\mathbb{P}$-a.s. to zero, which proves (3.6.3) for $\mathcal{C}^3_s(1,\rho)$.

3.6.4 Convergence of $\mathcal{C}^\varepsilon_s(1,\rho)$ and $\mathcal{C}^\varepsilon(1,\rho)$

In this section we prove Theorem 3.6. Let us first establish the claim for $\mathcal{C}^\varepsilon_s(1,\rho)$. For all $\varepsilon>0$, $(\theta_s\log\theta_s)^{1/2}\le\varepsilon a_s^{1/2}$, and so $\mathcal{C}^\varepsilon_s(1,\rho)\ge\mathcal{C}^2_s(1,\rho)$, which by Theorem 3.1.3 tends to $\mathrm{Asl}_\alpha(1/(1+\rho))$. As in Section 3.6.2 (Step 2) one can show that the upper bound tends to the same limit. It remains to prove the claim of Theorem 3.6 for $\mathcal{C}^\varepsilon(1,\rho)$. Let us write in short $\mathcal{A}^\varepsilon(\rho)$ for the event in the right hand side of (3.1.49), i.e. $\mathcal{C}^\varepsilon(1,\rho)=\mathcal{P}(\mathcal{A}^\varepsilon(\rho))$. One can show that

$$\mathcal{C}^{\varepsilon/2}_s(1-\varepsilon^2,\tfrac{\rho+2\varepsilon^2}{1-\varepsilon^2})-\left[1-\mathcal{C}^{\varepsilon/2}_s(1-\varepsilon^2,\tfrac{2\varepsilon^2}{1-\varepsilon^2})\right]-\delta_s\le\mathcal{C}^\varepsilon(1,\rho)\le\mathcal{C}^\varepsilon_s(1,\rho)+\delta_s, \tag{3.6.20}$$

where $\delta_s\equiv\delta_s(\rho,\varepsilon)$ is given by

$$\delta_s=\mathcal{P}(\max\nolimits_{v\in(1-\varepsilon^2,1+\rho+\varepsilon^2)}\max\nolimits_{v'\in(1-\varepsilon^2,1+\varepsilon^2)}|X_s(v')-X_s(v)|\le\varepsilon/2,(\mathcal{A}^\varepsilon(\rho))^c). \tag{3.6.21}$$

Now, by Theorem 1.3 in [2] (see the erratum [3] to this theorem) and Theorem 1.1 in [31], $\mathbb{P}$-a.s., $X_s\overset{J_1}{\Longrightarrow}Z_{d,\alpha}$. By definition of Skorohod's metric, there exists $\delta>0$ such that, for s large enough and $\lambda:[0,1+\rho]\to[0,1+\rho]$ strictly increasing and continuous,

$$\mathcal{P}\big(\max\{\max\nolimits_{v\in[0,1+\rho]}|X_s(\lambda(v))-Z_{d,\alpha}(v)|,\max\nolimits_{v\in[0,1+\rho]}|\lambda(v)-v|\}>\varepsilon^2\big)\le\delta, \tag{3.6.22}$$

which by the definition of Skorohod's metric vanishes as first $s\to\infty$ and then $\varepsilon\to 0$. By the statement of Theorem 3.1.6 for $\mathcal{C}^\varepsilon_s(1,\rho)$, $\mathcal{C}^\varepsilon_s(1,\rho)$ and $\mathcal{C}^{\varepsilon/2}_s(1-\varepsilon^2,\frac{\rho+2\varepsilon^2}{1-\varepsilon^2})$ tend to $\mathrm{Asl}_\alpha(1/(1+\rho))$ as first $s\to\infty$ and then $\varepsilon\to0$. It remains to show that $1-\mathcal{C}^{\varepsilon/2}_s(1-\varepsilon^2,\frac{2\varepsilon^2}{1-\varepsilon^2})$ vanishes. But this can be done as in Section 3.6.2 (Step 2). The proof of Theorem 3.6 is finished.

3.6.5 Convergence of $\mathcal{C}^i_s(1,\rho)$, $i=1,2,3$, for dynamics of BTM

Let us prove the claim of Theorem 3.8. For $i=1,2,3$ we establish that, $\mathbb{P}$-a.s., as $s\to\infty$,

$$|\mathcal{P}(\mathcal{A}^i_{s,\rho})-\mathcal{P}(\mathcal{A}_{s,\rho})|\to 0. \tag{3.6.23}$$

Let $i=1$. One can check that the proof of (3.6.23) can be carried out as in Section 3.6.1, once we have established the following. Namely, that, $\mathbb{P}$-a.s., for $Z^J_{s,k}$ such that $Z^J_{s,k}>\varepsilon$, for some $\varepsilon>0$

and some $k \leq k_s(M)$, there exists $x \in T_s$ such that $\ell_{k\theta_n}(x) - \ell_{(k-1)\theta_n}(x) > 0$. This can been as follows. Let $M > 0$ and $\varepsilon > 0$. The probability of the complement of this event is bounded by

$$\begin{aligned} & \mathcal{P}(\forall k \leq k_s(M) : Z^J_{s,k} > \varepsilon, \ell_{k\theta_n}(x) - \ell_{(k-1)\theta_n}(x) = 0 \forall x \in T_s) \\ \leq \; & \textstyle\sum_{k \leq k_s(M)} P(J(k\theta_n) = x)\mathcal{P}_x(Z^J_{s,k} > \varepsilon, \ell_{\theta_n}(x) = 0 \forall x \in T_s) \\ \leq \; & \textstyle\sum_{k \leq k_s(M)} P(J(k\theta_n) = x) \sum_y (s\varepsilon)^{-1}\mathcal{E}_x(\ell_{\theta_n}(y))\tau(y)\mathbb{1}_{\tau(y) \leq s\epsilon_s}. \end{aligned} \tag{3.6.24}$$

In expectation with respect to the random environment (3.6.24) is bounded by

$$(3.6.24) \leq \varepsilon^{-1}s^{-\alpha}\epsilon_s^{1-\alpha} \textstyle\sum_{k \leq k_n(M)} P(J(k\theta_n) = x) \sum_y E_x(\ell_{\theta_n}(y)). \tag{3.6.25}$$

Inserting (3.5.42) and (3.5.43), this is for all $d \geq 2$ smaller than $\varepsilon^{-1}M\epsilon_s^{(1-\alpha)/2}$. Thus, (3.6.24) tends $\mathbb{P}$-a.s. to zero as $N \to \infty$ and (3.6.23) follows for $i = 1$ as in Section 3.6.1. Now, let $i = 2$. The proof in Section 3.6.2 for BATM relies on Lemma 3.11 and therefore it suffices to prove the statement of Lemma 3.11 for J. But (3.5.8) implies (3.3.8) and (3.3.9) can be proved as Proposition 2.4.5 in [47]. Thus, the verification of (3.6.23) for $i = 2$ is along the lines of Section 3.6.2. Of course, having established the claim of Theorem 3.8 for $i = 1, 2$, the statement for $i = 3$ follows as in Section 3.6.3. The proof of Theorem 3.8 is finished.

3.7 Appendix

Proof of Lemma 3.11. Fix $x \in B_{a_n}$ and write $B = B_{r_n}(x)$. We use Proposition 2.18 in [4] to prove (3.3.8). This proposition states that there exists $c_4 \in (0, \infty)$ such that for all $m_n \gg r_n$ for which $U_z \leq m_n/r_n$ for all $z \in B$, $P_x(\eta(B) \leq m_n) \leq e^{-c_4 r_n^2 m_n^{-1}}$, as desired in (3.3.8). Since we assume $m_n \gg r_n$, it remains to verify whether $U_z \leq m_n/r_n$ for all $z \in B$. But $B \subseteq B_{2an}$ and (3.3.7) implies that, $\mathbb{P}$-a.s., $U_z \leq c_0(\log a_n)^3 \leq m_n/r_n$ for all $z \in B$. This finishes the proof of (3.3.8). The proof of (3.3.9) is as the proof of Lemma 3.2 in [31], where the claim is proved for $d = 2$.

□

Proof of Lemma 3.12. Since the proofs are the same for $x \in B_{a_n}$, we take for convenience $x = 0$. Let us first bound the contribution to (3.3.11) and (3.3.12) of y's that lie outside the ball $B_{\sqrt{m_n}\log m_n}$. By (3.3.8) of Lemma 3.11 we bound $\mathbb{P}$-a.s., $P(\sigma(y) \leq m_n) \leq e^{-c_4(\log|y|)^2}$ for $y \notin B_{\sqrt{m_n}\log m_n}$ and get

$$\begin{aligned} \textstyle\sum_{\sqrt{m_n}\log m_n \leq |y|} P(\sigma(y) \leq m_n) \;&\leq\; \textstyle\sum_{\sqrt{m_n}\log m_n \leq |y|} P(\sigma(y) \leq |y|^2/(\log|y|)^2) \\ &\leq\; \textstyle\sum_{\sqrt{m_n}\log m_n} |y|^{d-1}e^{-c_4(\log|y|)^2}, \end{aligned} \tag{3.7.1}$$

proving that the contribution of such y to the sums in (3.3.11) and (3.3.12) tends, $\mathbb{P}$-a.s. to zero. Now, let $y \in B_{\sqrt{m_n}\log m_n}$. The probability of $\sigma(y) \leq m_n$ is given by

$$\begin{aligned} P(\sigma(y) \leq m_n) \;&=\; P(\sigma(y) \leq m_n, \eta(B_n) \leq m_n) + P(\sigma(y) \leq m_n, \eta(B_n) > m_n) \\ &\leq\; P(\eta(B_n) \leq m_n) + P(\sigma(y) \leq \eta(B_n)), \end{aligned} \tag{3.7.2}$$

where $B_n \equiv B_n(0) \equiv B_{\sqrt{m_n}\log m_n}(0)$. By (3.3.8) of Lemma 3.11, the first probability in (3.7.2) is smaller than $e^{-c_4(\log m_n)^2}$. By the strong Markov property,

$$P(\sigma(y) \leq \eta(B_n)) = g_{B_n}(0, y)(g_{B_n}(y))^{-1}, \tag{3.7.3}$$

where $g_{B_n}(0, y) = E_0(\int_0^{\eta(B_n)} \mathbb{1}_{J(t)=y}dt)$. Write $B_n = A_1 \cup A_2$, where $A_1 = B_{\sqrt{m_n}/2}$, and $A_2 = B_n \setminus A_1$. We distinguish whether $d \geq 3$ or $d = 2$. Let first $d \geq 3$ and take $y \in A_1$. By

(3.3.2) and (3.3.4) of Theorem 3.10, $\mathbb{P}$-a.s., for all $y \in A_1$, $g_{B_n}(0,y) \leq c_3 \min(m_n^{-d/2+1}, |y|^{2-d})$, and so

$$\textstyle\sum_{y\in A_1} \mathbb{E}P(\sigma(y) \leq m_n) \leq \sum_{y \in A_1} c_3 \min(m_n^{-d/2+1}, |y|^{2-d}) \mathbb{E} g_{B_n}^{-1}(y). \tag{3.7.4}$$

For $y \in A_1$ we have $B_n \supseteq B_{\sqrt{m_n}}(y)$, implying that $g_{B_n}(y) \geq g_{B_{\sqrt{m_n}}(y)}(y)$. Together with the identical distribution of the τ's we get

$$(3.7.4) \leq \textstyle\sum_{y \in A_1} \mathbb{E} g^{-1}_{B_{\sqrt{m_n}}}(0) c_3 \min(m_n^{-d/2+1}, |y|^{2-d}) \leq c_6 m_n \mathbb{E} g^{-1}_{B_{\sqrt{m_n}}}(0) \leq c_5 m_n, \tag{3.7.5}$$

where we used (3.3.5) of Theorem 3.10 to bound $g^{-1}_{B_{\sqrt{m_n}}}(0) \leq U_0^{d-2}$ and used that the distribution of U_0 satisfies (3.3.3) of Theorem 3.10. Since $\sum_{y \in A_1} \min(m_n^{-d/2+1}, |y|^{2-d}) \leq \sqrt{m_n}$, the contribution coming from $y \in A_1$ is as claimed in (3.3.11). It remains to bound the contribution to (3.3.11) coming from A_2. Let $y \in A_2$. We bound $P(\sigma(y) \leq m_n)$ by

$$\begin{aligned} &\textstyle\sum_{z:\ |z|=|y|/2} P(\widetilde{J}(\eta(B_{|z|/2})) = z, \eta(B_{|y|/2}) \leq m_n, \sigma(y) \leq m_n) \\ &\leq \textstyle\sum_{z:\ |z|=|y|/2} P_z(\sigma(y) \leq m_n) P(\widetilde{J}(\eta(B_{|y|/2})) = z, \eta(B_{|y|/2}) \leq m_n). \end{aligned} \tag{3.7.6}$$

As in (3.7.2) and (3.7.3), $P_z(\sigma(y) \leq m_n) \leq 2c_3 |z-y|^{2-d} (g_{B_{\sqrt{m_n}}}(y))^{-1}$. But $2c_3|z-y|^{2-d} \leq c|y|^{2-d}$ for a suitable $c \in (0,\infty)$. Using this in (3.7.6) we see that

$$P(\sigma(y) \leq m_n) \leq \tfrac{c|y|^{2-d}}{g_{B_{\sqrt{m_n}}(y)}} P(\eta(B_{|y|/2}) \leq m_n) \leq \tfrac{c|y|^{2-d}}{g_{B_{\sqrt{m_n}}(y)}} \exp(-\tfrac{c_4}{4}|y|^2 m_n^{-1}), \tag{3.7.7}$$

where we used (3.3.8) of Lemma 3.11 in the last step. Calculating the sum over $y \in A_2$ of $\mathbb{E}P(\sigma(y) \leq m_n)$ we see that it is smaller than Cm_n. Moreover, the sum over $y \in A_2$ of $\mathbb{E}(P(\sigma(y) \leq m_n))^2$ is smaller than $C' m_n^{1/2}$. The proof of (3.3.11) is finished.

Let $d = 2$ and take $y \in A_1$. By (3.3.4) of Theorem 3.10 one can show that, $\mathbb{P}$-a.s., $g_{B_n}(0,y) \leq c_3(\log \sqrt{m_n}/|y|)$ for all $y \in A_1$, and so

$$\textstyle\sum_{y \in A_1} P(\sigma(y) \leq \theta_n) \leq \sum_{y \in A_1} c_3 g^{-1}_{B_n}(y) (\log \sqrt{m_n}/|y|). \tag{3.7.8}$$

By Lemma 3.3 in [31] we know that, $\mathbb{P}$-a.s., $g^{-1}_{B_n}(y) \leq g^{-1}_{B_{\sqrt{m_n}}(y)}(y) \leq (c_7 \log m_n)^{-1}$ for all $y \in A_1$. We set $f_{m_n}(|y|) \equiv 2c_3/c_7(1 - \log(|y|/\sqrt{m_n}))$. Calculating the sum over $y \in A_1$ of $f_{m_n}(y)$, we see that it is smaller than $c_5 m_n/(\log m_n)$. Also, the sum over $y \in A_2$ of $(f_{m_n}(y))^2$ is smaller than $c_5 m_n (\log m_n)^2$. Thus, contribution coming from $y \in A_1$ is as claimed in (3.3.12) for $k = 1, 2$. Let $y \in A_2$. As in (3.7.6) and (3.7.7) we bound

$$P_z(\sigma(y) \leq m_n) \leq 2g_{B_n(z)}(z,y)/g_{B_n(z)}(y) \leq 2g_{B_n(z)}(z,y)/g_{B_{\sqrt{m_n}}(y)}(y). \tag{3.7.9}$$

Since $|z-y| \geq \sqrt{m_n}/2$, one can check that $g_{B_n}(z,y) \leq c$ to obtain, $\mathbb{P}$-a.s., for all $y \in A_2$

$$P(\sigma(y) \leq m_n) \leq c \exp(-\tfrac{c_4}{4}|y|^2 m_n^{-1})(c_7 \log m_n)^{-1} \equiv f_{m_n}(|y|), \tag{3.7.10}$$

where we used that, $\mathbb{P}$-a.s., $g_{B_{\sqrt{m_n}}(y)}(y) \geq c_7 \log m_n$. The sum over $y \in A_2$ of $(f_{m_n}(|y|))^k$ satisfies (3.3.12) for $k = 1, 2$. This finishes the proof of Lemma 3.12. $\square$

Chapter 4

Aging beyond the arcsine law and super-aging in Bouchaud's asymmetric trap model on $\mathbb{Z}^d$

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Abstract

We study the impact of initial distributions on the behavior of correlation functions in the context of Bouchaud's asymmetric trap model on $\mathbb{Z}^d$, $d \geq 3$, and its symmetric version in $d = 2$. In particular, we establish for the class of initial distributions that were studied in [38] in the context of the random energy model that the probability of being at two time points in the same site exhibits super-aging. Moreover, we prove super-aging behavior of the probability of no jump in Bouchaud's asymmetric trap model on $\mathbb{Z}^d$, $d \geq 1$, for the same class of initial distributions. Our technique combines the method for convergence of correlation functions introduced in [37] with the method of convergence of clock processes on infinite graphs from [40].

4.1 Introduction and main results

We study the aging behavior of Bouchaud's asymmetric trap model on $\mathbb{Z}^d$. Bouchaud's trap models are simple phenomenological models that were introduced in [24, 26, 58, 59] to explain the slowdown of transients towards equilibrium of certain dynamics. Here, the dynamics is described through the motion of a particle that is crossing thermally activated barriers in a random energy landscape. Mathematically speaking, trap models are Markov jump processes, X, that evolve in random environments and are reversible with respect to a random Gibbs measure. Aging of X is quantified in the behavior of time-time correlation functions that measure the dependence of $X(t_w)$ on $X(t_w + t)$ for large t_w, t. The most prominent examples of such correlation functions are

$$R(t_w, t) \equiv \mathcal{P}_\mu(X(t_w) = X(t_w + t)), \quad \text{and,} \tag{4.1.1}$$

$$\Pi(t_w, t) \equiv \mathcal{P}_\mu(X(t_w) = X(t_w + t'),\ \forall 0 \leq t' \leq t_w), \qquad t, t_w > 0, \tag{4.1.2}$$

where $\mathcal{P}_\mu$ denotes the law of X with initial distribution μ. Physicists distinguish between normal aging, sub-aging and super-aging. A correlation function exhibits normal aging if it is for large waiting times a function of the ratio t/t_w. When it is for large waiting times a function t/t_w^δ, for $\delta \neq 1$, it exhibits sub-aging if $\delta < 1$, respectively super-aging if $\delta > 1$. Various models have been studied rigorously in the past years and aging behavior was proved for time-time correlation functions of the form (4.1.1) and (4.1.2); see for instance [11], [13], [17], [8], [37], [36] and [28] for spin glass models and [35], [14], [18], [16], and [40] for models on $\mathbb{Z}^d$.

Over the past decades a general mechanism was identified that leads to aging behavior: it is known that from convergence of the *clock process* (the total time that X spends along its trajectory) to an α-stable subordinator, $\alpha \in (0,1)$, one can derive the existence of an *arcsine aging* regime (see [17], [28], [37], and [40] for general criteria). More precisely, if the limit of the properly rescaled clock process is an α-stable subordinator then R, or Π, converge to the distribution function of the arcsine law,

$$\mathrm{Asl}_\alpha(u) = \frac{\sin \alpha\pi}{\pi} \int_0^u (1-x)^{-\alpha} x^{\alpha-1} dx, \quad 0 \le u \le 1. \tag{4.1.3}$$

Intuitively, as time evolves, X finds larger waiting times, so-called *traps*, which, observed on the right time scale, are in the domain of attraction of a stable law. However, also the choice of the *initial distribution* influences the aging behavior. The rôle of the initial distribution in the behavior of correlation functions was first made explicit in [37]. There, the most general limiting form of Π is identified using the renewal equation for (delayed) subordinators, where the delay is due to the contribution coming from the initial distribution. This idea was further exploited in [38], where it is established that a class of initial distributions delays the process in such a way that it exhibits super-aging. The method used in [38] is universal. However, the models in which it is applied are mean-field models. In this paper we implement the techniques of [37] and [38] in the context of Bouchaud's asymmetric trap model on $\mathbb{Z}^d$, $d \ge 3$, and its symmetric version on $\mathbb{Z}^2$.

Before going into further details, let us now define Bouchaud's asymmetric trap model on $\mathbb{Z}^d$, for $d \ge 1$. Let $G = (\mathcal{V}, \mathcal{L})$ be the d-dimensional lattice equipped with nearest-neighbor edges. Let $\left\{\tau(x), x \in \mathbb{Z}^d\right\}$ be a collection of i.i.d. random variables, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with continuous distribution function, and whose tail distribution is given by

$$\mathbb{P}(\tau(0) > u) = \begin{cases} Cu^{-\alpha}(1 + L(u)), & u \in (\bar{c}, \infty), \\ 1, & u \in (0, \bar{c}], \end{cases} \tag{4.1.4}$$

where $\alpha \in (0,1)$, $\bar{c}, C \in (0,\infty)$ are constants, and $L : (0,\infty) \to \mathbb{R}$ is a function that obeys $L(u) \to 0$ as $u \to \infty$. We refer to $\tau \equiv \left\{\tau(x), x \in \mathbb{Z}^d\right\}$ as the random environment. We write $x \sim y$ if x, y are nearest neighbors in $\mathbb{Z}^d$. Our process of interest, X, is a continuous time Markov jump process with initial distribution μ and jump rates $\lambda(x,y)$ that depend on a parameter, $\theta \in [0,1]$, and are given by

$$\lambda(x,y) = (\tau(x))^{\theta-1}(\tau(y))^{\theta}, \quad \text{if } x \sim y, \tag{4.1.5}$$

and zero else. One can check that X is reversible with respect to the measure that assigns to $x \in \mathbb{Z}^d$ a weight proportional to $\tau(x)$.

We introduce in this paper sufficient conditions for the initial distribution μ and a time scale c_ℓ for the correlation function $R(c_\ell t_w, c_\ell t)$ to converge, as $\ell \to \infty$, to

$$R_\infty(t_w, t) \equiv 1 - F(t_w + t) + \int_0^{t_w} \mathrm{Asl}_\alpha\left(\tfrac{t_w - v}{t_w + t - v}\right) dF(v), \tag{4.1.6}$$

where F is a distribution function on $[0,\infty)$ which might be random in the random environment. The same limiting form as in (4.1.6) is derived from a renewal equation for Π in [37]. It can be interpreted as follows. The first term, namely $1 - F$, is the probability that the contribution to Π coming from the initial distribution, σ, is larger than $(t_w + t)$ and the integral in R_∞ is the probability that the delayed subordinator, $\sigma + V_\alpha$, does not intersect $t_w, t_w + t$, conditioned on the event that σ is smaller than t_w. We then study the influence of the same initial distributions as in [38] on the aging behavior of this model. In particular, we prove the existence of a super-aging regime for R.

These initial distributions also are of the same form as the distribution of $X(t_w)$ in [58, 59], that is derived from the so-called partial equilibrium concept. This particular form of distribution allows *Bouchaud et al* to discuss the possible occurrence of a sub-aging regime in the asymmetric trap model on $\mathbb{Z}^d$, $d \geq 1$. This prediction is partially proved in [14], where it is established that the probability of no jump, i.e. Π, exhibits in Bouchaud's asymmetric trap model on $\mathbb{Z}$ sub-aging behavior for $\mu = \delta_0$, $0 \in \mathbb{Z}^d$. The exact value of the limiting correlation function Π in [14] differs from the one predicted in [58, 59], but both exhibit the same asymptotic behavior. In higher dimensions, the predictions of [58, 59] have not been established rigorously. The results of this paper show that the correlation functions R and Π exhibit on certain time scales super-aging and display the asymptotic behavior that was found in [58, 59].

4.1.1 Main results

We now state the main results of this paper. We begin with a theorem that establishes the limiting form of R for general initial distributions. The behavior of R has so far only been studied for $\mu = \delta_0$ in [40] where it is proved that $R(t_w, t_w\rho)$ converges, $\mathbb{P}$-a.s, as $t_w \to \infty$, to $\mathrm{Asl}_\alpha(1/(1+\rho))$. The emergence of the arcsine distribution relies on a scheme that is based on subordinators. The object that gives rise to this is the so-called clock process. There are two possible definitions of clocks: a discrete time clock process or a continuous time clock process. In [40], both types of clock processes are studied in a unified setting on infinite graphs. The result for Bouchaud's asymmetric trap model is derived from the convergence of a (properly rescaled) continuous time clock process. In this paper we study this clock process. To define it, let μ be a distribution on $\mathbb{Z}^d$. Let c_ℓ be diverging as $\ell \to \infty$. This is the time scale on which we observe the process X. Let J be a continuous time Markov chain with initial distribution μ and jump rates given by

$$\widetilde{\lambda}(x, y) \equiv \tau(x)^\theta \tau(y)^\theta, \quad \text{if } x \sim y, \tag{4.1.7}$$

and zero else. Let a_ℓ be a diverging sequence. This is an auxiliary time scale for J. Using this we define

$$S_\ell(t) \equiv c_\ell^{-1} \int_0^{\lfloor a_\ell t \rfloor} \tau(J(v))dv. \tag{4.1.8}$$

Note that then, $X(c_\ell t) = J(a_\ell S_\ell^{\leftarrow}(t))$, where $S_\ell^{\leftarrow}$ denotes the generalized right continuous inverse of S_ℓ. As in [40] we do not study S_ℓ directly, but a 'blocked' version of it. Let $1 \ll \theta_\ell \ll a_\ell$ be another diverging sequence. For $j \geq 0$ we define the block variables by

$$Z_{\ell,j} \equiv c_\ell^{-1} \int_{j\theta_\ell}^{(j+1)\theta_\ell} \tau(J(v))dv. \tag{4.1.9}$$

The first block variable, $Z_{\ell,0}$, plays a crucial rôle in the behavior of the correlation function. To emphasize this we separate the *initial block* from the sum and denote it by

$$\sigma_\ell \equiv Z_{\ell,0} = c_\ell^{-1} \int_0^{\theta_\ell} \tau(J(v))dv. \tag{4.1.10}$$

Then, we call the *pure clock* the rescaled blocked clock process defined through

$$S_\ell^b(t) \equiv \sum_{j=1}^{k_\ell(t)-1} Z_{\ell,j}, \quad t > 0, \tag{4.1.11}$$

where $k_\ell(t) \equiv \lfloor \lfloor a_\ell t \rfloor / \theta_\ell \rfloor$, and where by convention $\sum_{j=1}^{0} = 0$ so that $S_\ell(0) = 0$.

Let us now study the behavior of $\sigma_\ell + S_\ell^b$ for sequences of (possibly random) initial distributions μ having (possibly random) support $\mathcal{A}_\ell \equiv \{x \in \mathbb{Z}^d : \mu(x) > 0\}$ such that $|\mathcal{A}_\ell| < \infty$ for every finite ℓ. Note that the sequence$|\mathcal{A}_\ell|$ can be diverging as $\ell \to \infty$.

Due to the doubly-stochastic nature of $\sigma_\ell + S_\ell^b$ the question arises in which convergence mode with respect to $\mathbb{P}$ we can formulate statements. It turns out that the two relevant modes are either $\mathbb{P}$-almost sure convergence, or in $\mathbb{P}$-probability.

We are now ready to introduce sufficient conditions on μ, its support $\mathcal{A}_\ell$, and the time scale c_ℓ for the limit of $R(c_\ell t_w, c_\ell t)$ to be of the form (4.1.6). They are stated for fixed realization $\omega \in \Omega$ and given sequences c_ℓ.

(A-1) There exists a random variable σ having (possibly random) distribution function F on $[0, \infty)$ such that, for all $u \geq 0$,

$$\lim_{\ell \to \infty} \mathcal{P}_\mu(\sigma_\ell > u) = 1 - F(u). \tag{4.1.12}$$

(A-2) There exists $c \in (0, \infty)$ such that, for $\overline{\mathcal{A}}_\ell \equiv \{x \in \mathbb{Z}^d : \mathbb{P}(\mathcal{A}_\ell \supset x) > 0\}$,

$$\lim_{\ell \to \infty} |\overline{\mathcal{A}}_\ell|^{1/\alpha} c_\ell^{-1} (\log c_\ell \mathbb{1}_{d=2} + \mathbb{1}_{d \geq 3}) \leq c. \tag{4.1.13}$$

Theorem 4.1. *Let either $d \geq 3$ and $\theta \in [0, 1]$, or $d = 2$ and $\theta = 0$. Given $\gamma > 10$ set*

$$\begin{aligned} \theta_\ell &\equiv c_\ell^\alpha (\log c_\ell)^{-\gamma} \mathbb{1}_{d=2} + (\log c_\ell)^\gamma \mathbb{1}_{d \geq 3}, \\ a_\ell &\equiv c_\ell^\alpha (\log c_\ell)^{1-\alpha} \mathbb{1}_{d=2} + c_\ell^\alpha \mathbb{1}_{d \geq 3}. \end{aligned} \tag{4.1.14}$$

Assume that there exists an initial distribution μ and a diverging sequence c_ℓ such that Conditions (A-1)-(A-2) are verified $\mathbb{P}$-a.s. Then, $\mathbb{P}$-a.s., as $\ell \to \infty$,

(i) $S_\ell^b \Rightarrow V_\alpha$, where V_α is an α-stable subordinator, and where convergence holds weakly in the space $D[0, \infty)$ equipped with Skorohod's J_1 topology,

(ii) σ, viewed as an element of $D[0, \infty)$, is independent of V_α,

(iii) $R(c_\ell t_w, c_\ell t) \to R_\infty(t_w, t)$, where R_∞ is defined in (4.1.6).

If Conditions (A-1) and (A-2) hold in $\mathbb{P}$-probability, then assertions (i) and (ii) hold $\mathbb{P}$-a.s. and (iii) holds in $\mathbb{P}$-probability.

The choices of a_ℓ and θ_ℓ in Theorem 4.1 are, except for θ_ℓ in $d = 2$, the same as in [40]. In $d = 2$, the choice of θ_ℓ in (4.1.14) is slightly larger than in [40]. This is needed in order to establish that J does not revisit $J(0)$ after θ_ℓ.

Let us comment on the conditions of Theorem 4.1. Condition (A-1) implies, together with the first assertion of Theorem 4.1, that either $\mathbb{P}$-a.s. or in $\mathbb{P}$-probability, $\sigma_\ell + S_\ell^b \Rightarrow \sigma + V_\alpha$, as $\ell \to \infty$. Condition (A-2) links the time scale c_ℓ to the time that X spends in $\mathcal{A}_\ell$, or more precisely in $\overline{\mathcal{A}}_\ell$, where $\overline{\mathcal{A}}_\ell$ contains all vertices that have positive probability to belong to the support of μ. Heuristically, this can be seen as follows. Since the τ's are in the domain of attraction of an α-stable law, X spends most of its time in sites x for which $\tau(x)$ is maximal, which (since the τ's are i.i.d.) is of the order of $|\mathcal{A}_\ell|^{1/\alpha} \leq |\overline{\mathcal{A}}_\ell|^{1/\alpha}$. In $d = 2$, the process is recurrent and hence the time that X spends in $\mathcal{A}_\ell$, when we observe it on c_ℓ, is at most of the order of $|\overline{\mathcal{A}}_\ell|^{1/\alpha} \log c_\ell$. In higher dimensions it is transient and so the time it spends there is at most of the order of $|\overline{\mathcal{A}}_\ell|^{1/\alpha}$.

The pure clock is inherently scale invariant in the sense that it converges for all diverging time scales. By introducing an initial distribution whose size might be diverging, we perturb this behavior of the model. However, whenever σ is zero, the limiting clock is again scale invariant and Theorem 4.1 proves arcsine aging for X. We then say that the effect of the initial distribution is negligible. This is the case when the size of $\overline{\mathcal{A}}_\ell$ is independent of ℓ (see the special case $\mu = \delta_0$ in

[40]) , or when μ and c_ℓ are such that Condition (A-1) is satisfied for some c'_ℓ satisfying $c'_\ell \ll c_\ell$. Of course, if we start the process in a set of traps that it cannot escape on the time scale c_ℓ, then the process stays stranded in $\mathcal{A}_\ell$ and only σ emerges in the limit. We will see examples of this behavior below.

Remark. Notice that in Theorem 4.1 (and in all following theorems, except for Theorem 4.6) we assume for $d = 2$ that $\theta = 0$. We use this assumption to prove assertion (i) of Theorem 4.1 and we refer the reader for further explanations to Section 4.2.4.

A natural question that arises from Theorem 4.1 is how the distribution function F can affect the aging behavior of X, or whether there exist initial distributions that change the aging behavior of X. We now present a class of initial distributions that delay the process and hence yield to super-aging.

Let $\mu_{A,b}$ be a initial distribution with support $A \subset \mathbb{Z}^d$, and indexed by a parameter $b > 0$. Suppose that A is a finite subset of $\mathbb{Z}^d$ with size $|A| \equiv \ell$. For $b > \alpha$, we start X in A according to the initial distribution

$$\mu_{A,b}(x) = \frac{\tau(x)^b}{\sum_{y\in A}\tau(y)^b}, \quad x \in A. \tag{4.1.15}$$

The measures $\mu_{A,b}$ are well-understood (see e.g. [37]). It is known that they undergo a transition at the value $\alpha/b = 1$: if $\alpha/b > 1$ then $\sup_{x\in A}\mu_{A,b}(x)$ vanishes in $\mathbb{P}$-probability, as $\ell \to \infty$, and if $\alpha/b < 1$ then the extreme order statistics of $\{\mu_{A,b}(x), x \in A\}$ converges in $\mathbb{P}$-law, as $\ell \to \infty$, to a Poisson-Dirichlet measure. In the former one can show that the process exhibits arcsine aging and we are therefore here we interested in the latter case. Hence, distribution functions F that arise from initial distributions as in (4.1.15) are random in the random environment. Let us describe these distribution functions.

Let $\Gamma = \sum_{i=1}^{\infty}\delta_{\gamma_i}$ be a Poisson random measure with intensity measure $\nu_{\alpha/b}$, given by

$$\nu_{\alpha/b}(u,\infty) = u^{-\alpha/b}, \quad u > 0. \tag{4.1.16}$$

Given a parameter $\theta \in [0,1)$ and a sequence, $Y = \{Y_i, i \in \mathbb{N}\}$, of positive numbers define

$$F_{Y,\theta}(\rho) \equiv 1 - \sum_{j=1}^{\infty}\frac{\gamma_j}{\sum_{j=1}^{\infty}\gamma_j}\exp\Big(-\rho/\big(\gamma_j^{(1-\theta)/b}Y_j\big)\Big), \quad \rho \geq 0. \tag{4.1.17}$$

Let us now apply Theorem 4.1 to the initial distributions as in (4.1.15). Using the above mentioned Poisson-Dirichlet convergence one can establish convergence of σ_ℓ in $\mathbb{P}$-law, which is not strong enough for the application of Theorem 4.1. Therefore, we will not employ this theorem directly to σ_ℓ but exhibit an explicit representation of the random environment in which we can make almost sure convergence statements. This representation is taken from [37].

The time scale on which we observe the process is given by

$$c^{int}_{\ell,\delta} \equiv c_{\ell,\delta}\left(\log\ell\,\mathbb{1}_{d=2} + \mathbb{1}_{d\geq 3}\right), \tag{4.1.18}$$

where, for $\delta > 0$, we set

$$c_{\ell,\delta} \equiv \ell^{1/\alpha}t_w^{-\delta}. \tag{4.1.19}$$

For further references, we call $c^{int}_{\ell,\delta}$ the *intermediate time scale*. As a function of the size of A, this is the shortest time scale on which Condition (A-2) is satisfied.

Theorem 4.2. *Let either $d \geq 3$ and $\theta \in [0,1]$, or $d = 2$ and $\theta = 0$. There exists a collection of positive i.i.d. random variables, $Y \equiv Y(d,\theta,\alpha)$ that is independent of Γ and satisfies $\mathbb{E}(Y_1)^\kappa < \infty$, for all $\kappa \in \mathbb{R}$, such that for all $t_w > 0$ and all $t > 0$ we have*

$$\lim_{\ell\to\infty} R(c^{int}_{\ell,\delta}t_w, c^{int}_{\ell,\delta}t) = \mathcal{C}_{\infty,\delta}(t_w,t), \quad \textit{in } \mathbb{P}\textit{-law}, \tag{4.1.20}$$

where,

$$\mathcal{C}_{\infty,\delta}(t_w,t) \equiv 1 - F_{Y,0}\big(\tfrac{t_w+t}{t_w^\delta}\big) + \int_0^{t_w} \mathrm{Asl}_\alpha\big(\tfrac{t_w-v}{t_w-v+t}\big) dF_{Y,0}\big(\tfrac{v}{t_w^\delta}\big). \tag{4.1.21}$$

We derive from Theorem 4.2 three different regimes in the asymptotic behavior of $\mathcal{C}_{\infty,\delta}$ when t_w tends to infinity.

Corollary 4.3. *Let either $d \geq 3$ and $\theta \in [0,1]$, or $d = 2$ and $\theta = 0$. There exists a collection of i.i.d. positive random variables, $Y \equiv Y(d,\theta,\alpha)$, independent of Γ, such that for all $t_w > 0$ and all $t > 0$ we have,*
(i) for $\delta < 1$ and $\rho \geq 0$,

$$\lim_{\substack{t_w\to\infty \\ t/t_w^\delta=\rho}} \mathcal{C}_{\infty,\delta}(t_w,t) = 1, \quad \text{in } \mathbb{P}\text{-probability}, \tag{4.1.22}$$

(ii) for $\delta = 1$ and $\rho \geq 0$, in $\mathbb{P}$-law,

$$\lim_{\substack{t_w\to\infty \\ t/t_w^\delta=\rho}} \mathcal{C}_{\infty,\delta}(t_w,t) = \mathcal{C}_\infty^{full}(\rho) \equiv 1 - F_{Y,0}(1+\rho) + \int_0^1 \mathrm{Asl}_\alpha\big(\tfrac{1-v}{1-v+\rho}\big) dF_{Y,0}(v), \tag{4.1.23}$$

(iii) for $\delta > 1$ and $\rho \geq 0$,

$$\lim_{\substack{t_w\to\infty \\ t/t_w^\delta=\rho}} \mathcal{C}_{\infty,\delta}(t_w,t) = 1 - F_{Y,0}(\rho), \quad \text{in } \mathbb{P}\text{-law}. \tag{4.1.24}$$

From assertion (4.1.23) of Corollary 4.3 we derive aging behavior for $R(c_{\ell,1}^{int}t_w, c_{\ell,1}^{int}t)$, whereas (4.1.24) implies super-aging behavior for $R(c_{\ell,\delta}^{int}t_w, c_{\ell,\delta}^{int}t)$. It is evident from (4.1.24) that the aging behavior of R depends on $F_{Y,0}$. Before going into further discussion on the distribution function $F_{Y,0}$ let us prove that on longer time scales the process exhibits arcsine aging. This rejoins the above mentioned fact that the impact of the initial distribution becomes negligible on longer time scales.

For $m > 1$ we define the *long time scales* through

$$c_\ell^{lo} \equiv c_{\ell,1}^m \left(\log \ell \mathbb{1}_{d=2} + \mathbb{1}_{d\geq 3}\right). \tag{4.1.25}$$

Theorem 4.4. *Let either $d \geq 3$ and $\theta \in [0,1]$, or $d = 2$ and $\theta = 0$. For all $t_w > 0$ and all $t > 0$, $\mathbb{P}$-a.s.,*

$$\lim_{\ell\to\infty} R(c_\ell^{lo}t_w, c_\ell^{lo}t) = \mathrm{Asl}_\alpha\big(\tfrac{t_w}{t_w+t}\big). \tag{4.1.26}$$

Let us now discuss the distribution function $F_{Y,\theta}$, which is a slightly more general version of F than that obtained in [38] (where Y is a constant sequence). Let us study the asymptotic behavior of its expectation.

Theorem 4.5. *Let $Y = \{Y_i, i \geq 1\}$ be a collection of i.i.d. random variables that is independent of Γ. Suppose that either $\mathbb{E}Y_1^{(\alpha/(1-\theta))} < \infty$ and $\mathbb{E}Y_1^{-\min((b-\alpha)/(1-\theta),1)} < \infty$, or, Y_1^{-1} is in the domain of attraction of an α/θ-stable law, in short $Y^{-1} \in \mathcal{D}_{\alpha/\theta}$. Then, there exist $\kappa_i = \kappa_i(\theta, Y) \in (0,\infty)$, $i = 1,\ldots,4$, such that*

$$\lim_{\rho\to\infty} \rho^{\alpha/(1-\theta)}(1 - \mathbb{E}F_{Y,\theta}(\rho)) = \kappa_1, \tag{4.1.27}$$

$$\lim_{\rho\to 0} \rho^{-1}\mathbb{E}F_{Y,\theta}(\rho) = \kappa_2, \text{ if } b-\alpha > 1-\theta, \text{ and, if } Y^{-1} \in \mathcal{D}_{\alpha/\theta},\ \theta < \alpha, \tag{4.1.28}$$

$$\lim_{\rho\to 0} \rho^{-(b-\alpha)/(1-\theta)}\mathbb{E}F_{Y,\theta}(\rho) = \kappa_3, \text{ if } b-\alpha < 1-\theta, \text{ and, if } Y^{-1} \in \mathcal{D}_{\alpha/\theta},\ b\theta < \alpha, \tag{4.1.29}$$

$$\lim_{\rho\to 0} \rho^{-\alpha/\theta}\mathbb{E}F_{Y,\theta}(\rho) = \kappa_4, \text{ if } Y^{-1} \in \mathcal{D}_{\alpha/\theta},\ \theta > \alpha, \text{ and } b\theta > \alpha. \tag{4.1.30}$$

Notice that the two remaining combinations of (α, b, θ) *are* $\{b-\alpha > 1-\theta, \theta > \alpha, b\theta > \alpha\}$ *and* $\{b-\alpha < 1-\theta, \theta < \alpha, b\theta > \alpha\}$*, which are both empty sets.*

Distribution functions of similar type as $F_{Y,\theta}$ appeared previously in connection with aging. First, $F_{Y,\theta}$ is used in [58, 59] to explain the possible occurrence of a sub-aging regime for $\mathbb{E}\Pi(t_w, t)$ in Bouchaud's asymmetric trap model on $\mathbb{Z}^d$, $d \geq 1$. There, the elements of Y are given by

$$Y_i = (\textstyle\sum_{k=1}^{2d} \tau_{k,i}^{\theta})^{-1}, \quad i \geq 1, \tag{4.1.31}$$

where $\{\tau_{k,i}, k = 1, \dots 2d, i \in \mathbb{N}\}$ is an array of i.i.d. random variables, independent of Γ, and $\tau_{1,1}$ has the same distribution as $\tau(0)$. Then, it is established in [14] that $\mathbb{E}\Pi(t_w, t)$ converges in Bouchaud's asymmetric trap model on $\mathbb{Z}$ to a similar distribution function as $F_{Y,\theta}$. There, the weight associated with γ_j is given by $\gamma_j w_j$ where w_j is a different normalization than $\sum_{i\in\mathbb{N}} \gamma_i$. However, the asymptotic behavior of the expectation of the distribution function in [14] is the same as that of $\mathbb{E}F_{Y,\theta}$.

The correlation function that is studied by Bouchaud, Rinn, and Maass in [58, 59] is the probability of no jump, i.e. Π. This function displays, in the symmetric version of the model, the same behavior as R. However, in the asymmetric version Π differs substantially from R. Since the results on the continuous time clock process S_ℓ^b do not allow to draw consequence for Π on the same level of precision as in (4.1), we study Π only on *short time scales* on which the process is stranded in its starting point. They are given by

$$c_{\ell,\delta}^{sh} = c_{\ell,\delta}^{(1-\theta)/(1+\alpha)} \mathbb{1}_{d=1} + c_{\ell,\delta}^{(1-\theta)} \mathbb{1}_{d\geq 2}. \tag{4.1.32}$$

Theorem 4.6. *Let* $d \geq 1$ *and* $\theta \in [0, 1)$*. Let* $\delta > 0$ *be such that* $\delta > 1/(1-\theta)$ *if* $d \geq 2$*, and such that* $\delta > (1+\alpha)/(1-\theta)$ *if* $d = 1$*, and set* $\delta' = \delta/(1+\alpha)\mathbb{1}_{d=1} + \delta\mathbb{1}_{d\geq 2}$*. For* $Y = \{Y_i, i \geq 1\}$ *as in* (4.1.31) *we have*

$$\lim_{\substack{t_w\to\infty \\ t/t_w^{(1-\theta)\delta'}=\rho}} \lim_{\ell\to\infty} \Pi(c_{\ell,\delta}^{sh} t_w, c_{\ell,\delta}^{sh} t) = 1 - F_{Y,\theta}(\rho), \qquad \rho \geq 0, \quad \textit{in } \mathbb{P}\textit{-law}. \tag{4.1.33}$$

Since δ' is chosen in such a way that $\delta'(1-\theta) > 1$, Theorem 4.6 establishes super-aging for X in all dimensions.

Finally, we study another natural question, namely, whether there can be initial distributions that lead to several well-separated aging regimes for t_w. In fact, this is a simple consequence of Theorem 4.2 and Theorem 4.4 when the initial distribution is a convex combination of measures μ_{A_i,b_i}. Let us define this initial distribution. Fix $k \in \mathbb{N}$ and assume that A is a union of disjoint sets A_i for $i = 1, \dots, k$ satisfying,

$$\begin{aligned} |A_1| &= \lfloor \ell^{m_1} t_w^{\delta_1} \rfloor, \\ |A_i| &= \lfloor \ell^{m_i} t_w^{\delta_i} \rfloor, \quad i = 2, \dots, k, \end{aligned} \tag{4.1.34}$$

where $1 = m_1 \geq m_2 \geq \dots \geq m_k > 0$, and where $\delta_i > 0$ for $i = 1, \dots, k$. For $\lambda_1, \dots, \lambda_k$ such that $\lambda_i \geq 0$ for all i and $\sum_{i=1}^k \lambda_i = 1$ we define the initial distribution by

$$\mu_{A,b_1\dots,b_k}(x) = \textstyle\sum_{i=1}^k \lambda_i\, \mu_{A_i,b_i}(x), \qquad x \in A, \tag{4.1.35}$$

where for $b_i > \alpha$, μ_{A_i,b_i} is defined through (4.1.15). The time scale on which we observe the process is given by

$$\bar{c}_\ell = c_\ell(\log \ell \mathbb{1}_{d=2} + \mathbb{1}_{d\geq 3}). \tag{4.1.36}$$

In order to define the limit of $R(\bar{c}_\ell t_w, \bar{c}_\ell t)$, we construct random distribution functions $F_{Y,0}^i$ as follows. Let k_1 be the largest integer such that $m_{k_1} = 1$. For $i = 1, \dots, k_1$, let $\Gamma^i = \sum_{j=1}^\infty \delta_{\gamma_j^i}$ be a

Poisson random measure with intensity measure ν_{α/b_i} and let $Y^i = Y(d,\theta,\alpha) \equiv \{Y^i_j, j \in \mathbb{N}\}$ be a copy of Y as in Theorem 4.2. Assume that $\{\Gamma^i\}_{i=1}^{k_1}$ is a family of mutually independent Poisson random measures, $\{Y^i\}_{i=1}^{k_1}$ a family of i.i.d. collections, and that $\{\Gamma^i\}$ is independent of $\{Y^i\}$. Let $F^i_{Y,0}$ be the distribution function $F_{Y,0}$ when $\Gamma = \Gamma^i$ and $Y = Y^i$. We are now ready to determine the limit of $R(\bar{c}_\ell t_w, \bar{c}_\ell t)$.

Theorem 4.7. *Let either $d \geq 3$ and $\theta \in [0,1]$, or $d = 2$ and $\theta = 0$. For all $t_w > 0$ and all $t > 0$ we have*

$$\lim_{\ell\to\infty} R(\bar{c}_\ell t_w, \bar{c}_\ell t) = \mathcal{C}^{multi}_\infty(t_w,t), \qquad \text{in } \mathbb{P}\text{-law}, \tag{4.1.37}$$

where

$$\mathcal{C}^{multi}_\infty(t_w,t) \equiv \sum_{i=1}^{k_1} \lambda_i \mathcal{C}^i_{\infty,\delta_i}(t_w,t) + \mathrm{Asl}_\alpha\big(\tfrac{1}{1+t/t_w}\big) \sum_{i=k_1+1}^{k} \lambda_i, \tag{4.1.38}$$

and where $\mathcal{C}^i_{\infty,\delta_i}$ is as in (4.1.21) *associated to $\delta_i > 0$ and $F^i_{Y,0}$.*

One can derive from Theorem 4.7 and Corollary 4.3 that several choices of t, namely $t = t_w^{\delta_i}\rho$, $i = 1,\ldots,k$, lead to non-trivial limits of $\mathcal{C}^{multi}_\infty$ as t_w tends to infinity. In particular, one obtains this way that $\mathcal{C}^{multi}_\infty$ exhibits several different aging regimes.

The remainder of this paper is organized as follows. The first three sections are devoted to the proof of Theorem 4.1: assertions (i), (ii), and (iii) are proved in Section 4.2, Section 4.3, and Section 4.4, respectively. In Section 4.5 we introduce a representation of the random environment. Then, we study the initial block for initial distributions $\mu_{A,b}$ in Section 4.6. Section 4.7 contains proofs of all remaining theorems except Theorem 4.5. Properties of the random distribution function defined in (4.1.17) are studied in Section 4.8, proving in particular Theorem 4.5.

4.2 The pure clock process

In this section we prove assertion (i) of Theorem 4.1, that is we establish that if Condition (A-2) is verified then, $\mathbb{P}$-a.s., $S^b_\ell \Rightarrow V_\alpha$. Here and in what follows $\Rightarrow$ denotes weak convergence in the space $D[0,\infty)$ equipped with Skorohod's J_1 topology. The proof comes in two steps. First we establish that it suffices to study another process, $\overline{S}^b_\ell$, to which only those x contribute that are 'far enough' from $J(0)$ and for which $\tau(x)$ is 'large enough'. In the second step we prove that, $\overline{S}^b_\ell \Rightarrow V_\alpha$. For this, we give sufficient conditions for $\overline{S}^b_\ell$ to converge and then verify that these conditions are satisfied. These conditions are derived from Theorem 1.1 in [40] that establishes criteria for convergence of clock processes in the general setting of Markov jump processes on infinite graphs.

This section is organized as follows. In Section 4.2.1 we explain the steps of proof and list the key tools. We prove the results of Section 4.2.1 in Sections 4.2.2 and 4.2.3. Finally, Section 4.2.4 contains the verification of the above mentioned conditions for $\overline{S}^b_\ell$ to converge.

4.2.1 Key tools and strategy

Let us define the process $\overline{S}^b_\ell$. For this, we set

$$\epsilon_\ell = \epsilon_\ell(d) \equiv (\log c_\ell)^{-2}\mathbb{1}_{d=2} + (\log c_\ell)^{-3}\mathbb{1}_{d\geq 3}, \tag{4.2.1}$$

and denote by

$$\mathcal{T}_\ell \equiv \{x : \tau(x) > c_\ell \epsilon_\ell\}, \tag{4.2.2}$$

the collection of sites that carry 'large' traps and by $\mathcal{B}_\ell \equiv \mathbb{Z}^d \setminus \mathcal{T}_\ell$ its complement. We set

$$\overline{S}_\ell^b(t) \equiv \sum_{j=1}^{k_\ell(t)-1} \overline{Z}_{\ell,j}, \tag{4.2.3}$$

where,

$$\overline{Z}_{\ell,j} \equiv c_\ell^{-1} \sum_{x\in\mathbb{Z}^d\setminus B_{\overline{\theta}_\ell}(J(0))} \tau(x)\mathbb{1}_{x\in\mathcal{T}_\ell}(l_{(j+1)\theta_\ell}(x) - l_{j\theta_\ell}(x)), \tag{4.2.4}$$

and where we write $\overline{\theta}_\ell \equiv \theta_\ell^{1/2}(\log c_\ell)$. The following lemma states that $\overline{S}_\ell^b$ is a good approximation for S_ℓ.

Lemma 4.8. *Suppose that Condition (A-2) is satisfied. Then, $\mathbb{P}$-a.s.,*

$$\lim_{\varepsilon\to 0} \limsup_{\ell\to\infty} \mathcal{P}_\mu(\rho_\infty(\overline{S}_\ell^b, S_\ell^b) > \varepsilon) = 0, \tag{4.2.5}$$

where ρ_∞ denotes Skorohod's J_1 metric.

In order to establish that, $\mathbb{P}$-a.s., $\overline{S}_\ell^b \Rightarrow V_\alpha$ we follow a similar approach as in the proof of Theorem 1.4 in [40], where this claim is proved for $\mu = \delta_0$. There, the convergence of $\overline{S}_\ell^b$ is derived from Conditions (C-2)-(C-5) of Proposition 3.6 in [40]. This proposition is a specialization of Theorem 1.1 in [40] to the setting of Bouchaud's asymmetric trap model. We now present a version of this proposition adapted to general initial distributions μ. For this, we introduce the following quantities. For $t > 0$, set

$$\pi_\ell^{\mu,t}(x) \equiv (k_\ell(t))^{-1} \sum_{k=1}^{k_\ell(t)-1} P_\mu(J(k\theta_\ell) = x), \quad x \in \mathbb{Z}^d. \tag{4.2.6}$$

Moreover, for $x, y \in \mathbb{Z}^d$, $u > 0$, and $\varepsilon > 0$ define

$$Q_\ell^u(x,y) \equiv \mathcal{P}_x\left(\ell_{\theta_\ell}(y)\tau(y) > c_\ell u, \eta(B_{\overline{\theta}_\ell}(x)) > \theta_\ell\right)\mathbb{1}_{y\in\mathcal{T}_\ell}, \tag{4.2.7}$$

$$M_\ell^\varepsilon(x,y) \equiv c_\ell^{-1}\mathcal{E}_x\left(\ell_{\theta_\ell}(y)\tau(y)\mathbb{1}_{\tau(y)\ell_{\theta_\ell}(y)\le c_\ell\varepsilon}\mathbb{1}_{\eta(B_{\overline{\theta}_\ell}(x))>\theta_\ell}\right)\mathbb{1}_{y\in\mathcal{T}_\ell}, \tag{4.2.8}$$

where for a set $B \subset \mathbb{Z}^d$ we write $\eta(B) = \inf\{t > 0 : J(t) \notin B\}$ for its exit time, and to simplify notation we write $\mathcal{P}_x \equiv \mathcal{P}_{\delta_x}$ for the law of X. Our conditions bear on the following key objects

$$\widetilde{\nu}_\ell^{\mu,t}(u,\infty) \equiv k_\ell(t) \sum_{z\in\overline{\mathcal{A}}_\ell} \mu(z) \sum_{x\in B_{d_\ell(t)}(z)\setminus B_{\overline{\theta}_\ell}(z)} \pi_\ell^{\delta_z,t}(x) \sum_{y\in\mathbb{Z}^d} Q_\ell^u(x,y), \tag{4.2.9}$$

$$\widetilde{\sigma}_\ell^{\mu,t}(u,\infty) \equiv k_\ell(t) \sum_{z\in\overline{\mathcal{A}}_\ell} \mu(z) \sum_{x\in B_{d_\ell(t)}(z)\setminus B_{\overline{\theta}_\ell}(z)} \pi_\ell^{\delta_z,t}(x) \sum_{y\in\mathbb{Z}^d} (Q_\ell^u(x,y))^2, \tag{4.2.10}$$

$$m_\ell^{\mu,t}(\varepsilon) \equiv k_\ell(t) \sum_{z\in\overline{\mathcal{A}}_\ell} \mu(z) \sum_{x\in B_{d_\ell(t)}(z)\setminus B_{\overline{\theta}_\ell}(z)} \pi_\ell^{\delta_z,t}(x) \sum_{y\in\mathbb{Z}^d} M_\ell^\varepsilon(x,y), \tag{4.2.11}$$

where for $t > 0$, we write $d_\ell(t) \equiv (a_\ell t)^{1/2} \log a_\ell t$. We are now ready to introduce our conditions. They are stated for fixed $x \in \mathbb{Z}^d$, $u > 0$, $t > 0$, $\varepsilon > 0$, and $\omega \in \Omega$.

(C-2') $\lim_{\ell\to\infty} \widetilde{\nu}_\ell^{\mu,t}(u,\infty) = t\mathcal{K}u^{-\alpha}$.

(C-3') $\lim_{\ell\to\infty} \widetilde{\sigma}_\ell^{\mu,t}(u,\infty) = 0$.

(C-4') $\limsup_{\ell\to\infty} m_\ell^{\mu,t}(\varepsilon) \le C(t)\varepsilon^{1-\alpha}$.

Proposition 4.9. *Suppose that for an initial distribution μ, for all $u > 0$, $t > 0$, and $\varepsilon > 0$ there exists $\Omega^\circ = \Omega^\circ(u,t,\varepsilon)$ with $\mathbb{P}(\Omega^\circ) = 1$ and such that (C-2') - (C-4') are satisfied for all $\omega \in \Omega^\circ$. Then, $\mathbb{P}$-a.s., $\overline{S}_\ell^b \Rightarrow V_\alpha$.*

We conclude this section with remarks on the methods that allow us to prove Lemma 4.8 and Proposition 4.9.

Let us first compare (C-2')-(C-4') with (C-2)-(C-5) of Proposition 3.6 in [40]. The quantities of interest in (C-2)-(C-4) can be obtained from $\widetilde{\nu}_\ell^t(u,\infty)(z)$, $\widetilde{\sigma}_\ell^t(u,\infty)(z)$, and $m_\ell^t(\varepsilon)(z)$ when one substitutes δ_0 for μ and $\mathbb{E}\pi_\ell^{\delta_0,t}(z,x)$ for $\pi_\ell^{\delta_0,t}(z,x)$. This difference is, apart from the initial distribution, due to the fact that there is no analogue to (C-5) of Proposition 3.6 in [40] for general μ. This condition relies on a $\mathbb{P}$-a.s. local central limit theorem for J (Theorem 5.14 in [4]), which is stated for initial distributions of the form δ_x, $x \in \mathbb{Z}^d$. Since the proof of this theorem uses an ergodic theorem for which no convergence speed is known it cannot not be generalized to initial distributions with diverging support.

The proofs of Lemma 4.8 and Proposition 4.9 heavily use on a result for the heat kernel, $P_{\delta_x}(J(t) = y)$, of J that is taken from [4]. For further references, let us restate this theorem. To simplify notation, let us write for $x \in \mathbb{Z}^d$, $P_x \equiv P_{\delta_x}$ for the law of J.

Theorem 4.10 (Theorem 1.2 in [4]). *There exist identically distributed random variables $\{U_x\}_{x\in\mathbb{Z}^d}$ such that*

$$\mathbb{P}(U_x > v) \le c_1 \exp(-c_2 v^{1/3}), \quad v > 0, \tag{4.2.12}$$

where $c_1, c_2 \in (0,\infty)$, and such that for all fixed $\omega \in \Omega$ the following holds. For all $x, y \in \mathbb{Z}^d$, and $t > 0$,

$$P_x(J(t) = y) \le c_1 t^{-d/2}, \tag{4.2.13}$$

and, if furthermore, $|x-y| \vee t^{1/2} \ge U_x$,

$$P_x(J(t) = y) \le c_1 t^{-d/2} e^{-c_2|x-y|\{1\wedge|x-y|t^{-1}\}}. \tag{4.2.14}$$

In order to use Theorem 4.10 the following bounds, taken from Lemma 3.3 in [2], are helpful. There exist $c_0 \in (0,\infty)$ and $n_0 \in \mathbb{N}$ such that, $\mathbb{P}$-a.s., for $n \ge n_0$ we have for all $B \subset \mathbb{Z}^d$ satisfying $\log|B| = O(\log n)$,

$$\sup_{x:\, x\in B} U_x \le c_0(\log|B|)^3. \tag{4.2.15}$$

This implies that, $\mathbb{P}$-a.s., for $c_\ell \ge n_0$, $U_x \le c_0(\log c_\ell)^3$ for all $x \in \bigcup_{z\in\overline{\mathcal{A}}_\ell} B_{a_\ell}(z)$. Hence, whenever we apply (4.2.14) of Theorem 4.10 we check whether, given $x, y \in \bigcup_{z\in\overline{\mathcal{A}}_\ell} B_{a_\ell}(z)$ and $t > 0$, $|x-y| \wedge t^{1/2} \gg (\log c_\ell)^3$.

Finally, let us remark that, in order to establish the statement of Theorem 4.1 for all increasing sequences c_ℓ, we first consider subsequences of the form $c_\ell = e^{(r+n)^k}$ for $r \in [0,1]$, where k is chosen such that for all $r \in [0,1]$,

$$(\log c_\ell)^{-(b-\alpha)/2} + (\log c_\ell)^{-(1-\alpha)} + (\log c_\ell)^{-\alpha/2} \le n^{-2}. \tag{4.2.16}$$

We then prove that, for all $r \in [0,1]$ there exists a sequence $\Omega_n(r)$ such that $\mathbb{P}((\Omega_n(r))^c)$ is smaller than the left hand side of (4.2.16) and such that, on $\Omega(r) = \bigcap_{n\in\mathbb{N}} \Omega_n(r)$, the claim of Theorem 4.1 holds. By Lemma 3.1 in [40] this implies that the claim is true for general c_ℓ. Indeed, since the bounds on $\mathbb{P}(\Omega_n(r))$ are independent of r, we know by Lemma 2.1 in [40] that $\mathbb{P}(\Omega^\circ) = 1$ for $\Omega^\circ = \bigcap_{r\in[0,1]} \Omega(r)$. Therefore we take $r = 0$.

4.2.2 Proof of Lemma 4.8

By definition of Skorohod's J_1 metric, it suffices to establish the claim of Lemma 4.8 for ρ_∞ replaced by ρ_r, Skorohod's J_1 metric on $D[0,r]$ for all $r > 0$. For convenience we take $r = 1$ and get

$$\mathcal{P}_\mu(\rho_1(S_\ell^b, \overline{S}_\ell^b) > \varepsilon) \le \mathcal{P}_\mu\left(\textstyle\int_0^{a_\ell} \tau(J(s))\mathbb{1}_{J(s)\in\mathcal{B}_\ell} ds + \int_{\theta_\ell}^{a_\ell} \tau(J(s))\mathbb{1}_{J(s)\in B_{\overline{\theta}_\ell}(J(0))} > c_\ell\varepsilon\right) \tag{4.2.17}$$

We prove now that, $\mathbb{P}$-a.s.,

$$\lim_{\ell\to\infty} \mathcal{P}_\mu\left(\int_0^{a_\ell} \tau(J(s))\mathbb{1}_{\tau(J(s))\in\mathcal{B}_\ell} > c_\ell \epsilon_\ell^{(1-\alpha)/3}\right) = 0, \tag{4.2.18}$$

and that

$$\lim_{\ell\to\infty} \mathcal{P}_\mu\left(\int_{\theta_\ell}^{a_\ell} \tau(J(s))\mathbb{1}_{J(s)\in B_{\bar{\theta}_\ell}(J(0))\cap\mathcal{T}_\ell} > c_\ell\varepsilon\right) = 0, \quad \forall\varepsilon > 0. \tag{4.2.19}$$

By a first order Chebyshev inequality, the probability in (4.2.18) is bounded above by

$$c_\ell^{-1}\epsilon_\ell^{-(1-\alpha)/3} \sum_x \mu(x) \sum_y E_x(l_{a_\ell}(y))\tau(y)\mathbb{1}_{y\in\mathcal{B}_\ell}. \tag{4.2.20}$$

Let us first construct a bound on $E_x(l_{a_\ell}(y))$, valid $\mathbb{P}$-a.s. and uniformly in $x \in \overline{\mathcal{A}}_\ell$. Given $m_\ell \geq \theta_\ell$, and $c \in (0,\infty)$, let $e_{m_\ell}(s)$, $s \geq 0$, be the function defined through

$$e_{m_\ell}(s) \equiv c \begin{cases} \mathbb{1}_{d\geq 3} + \log m_\ell \mathbb{1}_{d=2}, & s \leq \theta_\ell^{1/(2d)}, \\ s^{2-d}\mathbb{1}_{d\geq 3} + \log m_\ell \mathbb{1}_{d=2}, & \theta_\ell^{1/(2d)} \leq s \leq \left(\frac{m_\ell}{\log m_\ell}\right)^{1/2}, \\ s^{2-d}\mathbb{1}_{d\geq 3} + \log\log m_\ell \mathbb{1}_{d=2}, & \left(\frac{m_\ell}{\log m_\ell}\right)^{1/2} \leq s \leq m_\ell^{1/2} \log\log c_\ell, \\ (s^{2-d}\mathbb{1}_{d\geq 3} + \mathbb{1}_{d=2})e^{-c_4 s^2/m_\ell} & \text{else} . \end{cases} \tag{4.2.21}$$

By Theorem 4.10 and Lemma 3.2 in [40] we have, $\mathbb{P}$-a.s.,

$$E_x(l_{a_\ell}(y)) \leq e_{a_\ell}(|x-y|), \quad \forall x, y \in \mathbb{Z}^d. \tag{4.2.22}$$

Therefore (4.2.20) is, $\mathbb{P}$-a.s., bounded above by

$$c_\ell^{-1}\epsilon_\ell^{-(1-\alpha)/3} \sum_x \mu(x) \sum_y e_{a_\ell}(|x-y|)\tau(y)\mathbb{1}_{y\in\mathcal{B}_\ell}. \tag{4.2.23}$$

Let us now distinguish whether $x \in \overline{\mathcal{A}}_\ell \cap \mathcal{T}_\ell$ or $x \in \overline{\mathcal{A}}_\ell \cap \mathcal{B}_\ell$. For this, we bound

$$\mu(x) \leq \mu_1(x) + \mu_2(x) \equiv \mu(x|\overline{\mathcal{A}}_\ell \cap \mathcal{T}_\ell) + \mu(x|\overline{\mathcal{A}}_\ell \cap \mathcal{B}_\ell), \tag{4.2.24}$$

and control the contribution to (4.2.23) coming from μ_1 and μ_2 separately. We call (I) the contribution coming from μ_1 and (II) that from μ_2. Let us begin with (I). Since $\mu_1(x)$ and $\tau(y)\mathbb{1}_{y\in\mathcal{B}_\ell}$ are independent,

$$\mathbb{E}(I) \leq c_\ell^{-1}\epsilon_\ell^{-(1-\alpha)/3} \sum_{x\in\mathbb{Z}^d} \mathbb{E}[\mu_1(x)]\mathbb{E}[\tau(0)\mathbb{1}_{0\in\mathcal{B}_\ell}] \sum_{y\in\mathbb{Z}^d} e_{a_\ell}(|y|). \tag{4.2.25}$$

Now, $\sum_y e_{a_\ell}(|y|) \leq a_\ell(\log\log c_\ell)^2$. Moreover, on can check that $\mathbb{E}\tau(0)\mathbb{1}_{0\in\mathcal{B}_\ell} \leq c(c_\ell\epsilon_\ell)^{1-\alpha}$. Using the fact that μ_1 is a probability measure,

$$\mathbb{E}(I) \leq \epsilon_\ell^{-(1-\alpha)/3} a_\ell(\log\log c_\ell)^2 c_\ell^{-\alpha}\epsilon_\ell^{1-\alpha}, \tag{4.2.26}$$

which vanishes as $\ell \to \infty$ by (4.1.14). Now, let us control (II). We distinguish whether $y \in \overline{\mathcal{A}}_\ell$ or $y \notin \overline{\mathcal{A}}_\ell$. When $y \notin \overline{\mathcal{A}}_\ell$, then $\mu_2(x)$ is independent of $\tau(y)$ and we can proceed as in (4.2.25). For $y \in \overline{\mathcal{A}}_\ell$ we bound $e_{a_\ell}(|y|) \leq \log a_\ell \mathbb{1}_{d=2} + \mathbb{1}_{d\geq 3}$ and get that

$$\begin{aligned} & c_\ell^{-1}\epsilon_\ell^{-(1-\alpha)/3} \sum_{x\in\mathbb{Z}^d} \sum_{y\in\mathbb{Z}^d} \mathbb{E}[\mu_2(x)\tau(y)\mathbb{1}_{y\in\mathcal{B}_\ell\cap\overline{\mathcal{A}}_\ell}]e_{a_\ell}(|y|) \\ \leq \; & \epsilon_\ell^{-(1-\alpha)/3}|\overline{\mathcal{A}}_\ell| c_\ell^{-\alpha}\epsilon_\ell^{(1-\alpha)}(\log a_\ell \mathbb{1}_{d=2} + \mathbb{1}_{d\geq 3}). \end{aligned} \tag{4.2.27}$$

This tends to zero by Condition (A-2). The claim of (4.2.18) now follows from Borel-Cantelli Lemma.

Let us now prove (4.2.19). Since the probability therein is decreasing in $\varepsilon > 0$ it suffices to consider fixed $\varepsilon > 0$. Then, it is bounded from above by

$$\mathcal{P}_\mu\left(\int_0^{a_\ell} \tau(J(s))\mathbb{1}_{J(s)\in B_{\overline{\theta}_\ell}(J(0))\cap\mathcal{T}_\ell} > c_\ell\varepsilon | J(0)\in\mathcal{B}_\ell\right) \tag{4.2.28}$$

$$+\ \mathcal{P}_\mu\left(\int_{\theta_\ell}^{a_\ell} \tau(J(s))\mathbb{1}_{J(s)\in B_{\overline{\theta}_\ell}(J(0))\cap\mathcal{T}_\ell} > c_\ell\varepsilon | J(0)\in\mathcal{T}_\ell\right). \tag{4.2.29}$$

We show now that (4.2.28) and (4.2.29) vanish $\mathbb{P}$-a.s. By (4.2.18) and definition of μ_2, (4.2.28) is, $\mathbb{P}$-a.s., for c_ℓ large enough, bounded above by

$$\begin{aligned} &\varepsilon + \textstyle\sum_{y\in\mathcal{B}_\ell}\mu_2(y)\mathcal{P}_y(\exists y\in\mathcal{T}_\ell\cap B_{\overline{\theta}_\ell}(y) : l_{a_\ell}(x) > 0) \\ \leq\ &\varepsilon + \textstyle\sum_{y\in\mathcal{B}_\ell}\sum_{x\in B_{\overline{\theta}_\ell}(y)}\mu_2(y)\mathbb{1}_{x\in\mathcal{T}_\ell}. \end{aligned} \tag{4.2.30}$$

Since μ_2 is independent of $\mathbb{1}_{x\in\mathcal{T}_\ell}$,

$$\textstyle\sum_{y\in\mathbb{Z}^d}\sum_{x\in B_{\overline{\theta}_\ell}(y)}\mathbb{E}(\mu_2(y)\mathbb{1}_{x\in\mathcal{T}_\ell}) \leq \overline{\theta}_\ell^d c_\ell^{-\alpha}\epsilon_\ell^{-\alpha} \leq (\log\theta_\ell)^{-\gamma/2}\mathbb{1}_{d=2} + c_\ell^{-\alpha/2}\mathbb{1}_{d\geq 3}, \tag{4.2.31}$$

which tends to zero. Thus, by Borel-Cantelli Lemma, (4.2.28) vanishes $\mathbb{P}$-a.s. Let us now establish that (4.2.29) tends to zero. For this, we first calculate the probability that there exists $x\in\mathcal{T}_\ell\setminus\overline{\mathcal{A}}_\ell$ such that $x\in B_{\overline{\theta}_\ell}(\overline{\mathcal{A}}_\ell\cap\mathcal{T}_\ell)$, where for a set B we write $B_{\overline{\theta}_\ell}(B)\equiv\bigcup_{x\in B}B_{\overline{\theta}_\ell}(x)$. We have that

$$\mathbb{P}(\exists x\in\mathcal{T}_\ell\cap B_{\overline{\theta}_\ell}(\overline{\mathcal{A}}_\ell\cap\mathcal{T}_\ell) :\ x\notin\overline{\mathcal{A}}_\ell) \leq \mathbb{E}[\textstyle\sum_{x\in\overline{\mathcal{A}}_\ell}\mathbb{1}_{x\in\mathcal{T}_\ell}\sum_{y\in B_{\overline{\theta}_\ell}(x)\setminus\{x\}}\mathbb{1}_{y\in\mathcal{T}_\ell}], \tag{4.2.32}$$

which is bounded above by $|\overline{\mathcal{A}}_\ell|(c_\ell\epsilon_\ell)^{-2\alpha}\overline{\theta}_\ell^d$. By Condition (A-2) this tends to zero. Thus, $\mathbb{P}$-a.s., $|B_{\overline{\theta}_\ell}(\overline{\mathcal{A}}_\ell\cap\mathcal{T}_\ell)| = \mathcal{T}_\ell\cap\overline{\mathcal{A}}_\ell$. In order prove (4.2.29) it remains to establish that

$$\textstyle\sum_{x\in\overline{\mathcal{A}}_\ell\cap\mathcal{T}_\ell}\mu(x)\mathcal{P}_x(c_\ell^{-1}\tau(x)(\ell_{a_\ell}(x)-\ell_{\theta_\ell}(J(x)) > \varepsilon) \tag{4.2.33}$$

vanishes $\mathbb{P}$-a.s. Suppose first that $c_\ell^{-1}\tau(x)\leq(\log c_\ell)^{1/2}$. Then, each summand in (4.2.33) is smaller than

$$P_x(l_{a_\ell}(x)-l_{\theta_\ell}(x) > \delta(\log\theta_\ell)) \leq 2/\delta(\log c_\ell)^{-1/2}E_x[l_{a_\ell}(x)-l_{\theta_\ell}(x)] \leq \kappa_\ell, \tag{4.2.34}$$

where $\kappa_\ell\equiv(\log c_\ell)^{-1/2}(\log\log a_\ell\mathbb{1}_{d=2}+\theta_\ell^{-d/2+1}\mathbb{1}_{d\geq 3})$ and where we used (4.2.13) of Theorem 4.10 to bound $E_x[l_{a_\ell}(x)-l_{\theta_\ell}(x)]$. Now, by Condition (A-2) and (4.1.14),

$$\kappa_\ell\textstyle\sum_{x\in\overline{\mathcal{A}}_\ell}\mathbb{E}[\mathbb{1}_{x\in\mathcal{T}_\ell}] \ll (\log c_\ell)^{-1+\alpha}\mathbb{1}_{d=2}+\theta_\ell^{-1/3}\mathbb{1}_{d\geq 3}, \tag{4.2.35}$$

which vanishes as $\ell\to\infty$. It remains to bound the probability that $\max_{x\in\overline{\mathcal{A}}_\ell}c_\ell^{-1}\tau(x) > (\log c_\ell)^{1/2}$. We have that,

$$\textstyle\sum_{x\in\overline{\mathcal{A}}_\ell}\mathbb{P}(\tau(x) > c_\ell(\log c_\ell)^{1/2}) \leq |\overline{\mathcal{A}}_\ell|c_\ell^{-\alpha}(\log c_\ell)^{-\alpha/2}, \tag{4.2.36}$$

which by Condition (A-2), is bounded above by $(\log c_\ell)^{-\alpha/2}$. Thus, (4.2.29) vanishes by Borel-Cantelli Lemma, $\mathbb{P}$-a.s. The proof of Lemma 4.8 is complete.

4.2.3 Proof of Proposition 4.9

We establish that (C-2')-(C-4') imply the Conditions (A-1)-(A-4) of Theorem 1.1 in [40]. As in the proof of Proposition 3.6 in [40], the first Condition follows from (4.2.13) of Theorem 4.10. Thus, it suffices to prove that (C-2')-(C-4') imply (A-2)-(A-4). There are only two difference between the quantities in Theorem 1.1 in [40] and those in (4.2.9)-(4.2.11). The first is that in (4.2.9)-(4.2.11) the sum over x is restricted to $x \in B_{d_\ell(t)}(z) \setminus B_{\overline{\theta}_\ell}(z)$ instead of $x \in \mathbb{Z}^d$. The restriction to $\mathbb{Z}^d \setminus B_{\overline{\theta}_\ell}(z)$ follows from the definition of the $Z_{n,i}$'s and the restriction to $B_{d_\ell(t)}(z)$ follows from (3.8) of Lemma 3.2 in [40] which states that, for all starting points $z \in \overline{\mathcal{A}}_\ell$, J does not exit $B_{d_\ell(t)}(z)$ with probability larger than $1 - \exp(-c_4^2(\log a_\ell))$. The second difference between our quantities and those in Theorem 1.1 in [40] lies in the Q_ℓ^u's and M_ℓ^ε's. To establish that (C-2')-(C-4') $\Rightarrow$ (A-2)-(A-4) we prove that

$$\begin{aligned}(I) + (II) \equiv \textstyle\sum_{z\in\overline{\mathcal{A}}_\ell} \sum_{x\in B_{d_\ell(t)}(z)} \pi_\ell^{t,\delta_z}(x)|\mathcal{P}_x(\overline{Z}_{\ell,1} > u) - \sum_y Q_\ell^u(x,y)| \\ + \textstyle\sum_{z\in\overline{\mathcal{A}}_\ell} \sum_{x\in B_{d_\ell(t)}(z)} \pi_\ell^{t,\delta_z}(x)|\mathcal{E}_x\overline{Z}_{\ell,1}\mathbb{1}_{\overline{Z}_{\ell,1}\le\varepsilon} - \sum_y M_\ell^\varepsilon(x,y)|,\end{aligned} \tag{4.2.37}$$

tends $\mathbb{P}$-a.s. to zero. In order to establish that (I) vanishes, we prove that, $\mathbb{P}$-a.s.,

$$\lim_{\ell\to\infty} \textstyle\sum_{z\in\overline{\mathcal{A}}_\ell} \mu(z) \sum_{x\in B_{d_\ell(t)}(z)\setminus B_{\overline{\theta}_\ell}(z)} \pi_\ell^{\delta_z,t}(x)\mathcal{P}_x(B_1) = 0, \quad \forall t > 0, \tag{4.2.38}$$

where $B_1 \equiv \{\#\{x : x \in \mathcal{T}_\ell, l_{\theta_\ell}(x) > 0\} \ge 2\}$. The proof of (4.2.38) follows a similar scheme as the proof of Lemma 4.8. We first construct a deterministic bound, $\widetilde{\pi}_\ell^1(z-x)$, on $\pi_\ell^{\delta_z,1}(x)$. For the construction of $\widetilde{\pi}_\ell^1(z-x)$, we distinguish whether $d = 2$ or $d \ge 3$. For $d = 2$, J is by assumption not random in the environment, and so we take $\widetilde{\pi}_\ell^1(z-x) \equiv \pi_\ell^{1,\delta_z}(x)$. For $d \ge 3$, $y \in \mathbb{Z}^d$ and $t > 0$ we set

$$\widetilde{\pi}_\ell^t(y) \equiv (k_\ell(t)\theta_\ell)^{-1}c_3|y|^{2-d}(\mathbb{1}_{|y|\le\sqrt{a_\ell t}\log\theta_\ell} + e^{-1/2|y|^2/a_\ell}\mathbb{1}_{|y|>\sqrt{a_\ell t}\log\log a_\ell}). \tag{4.2.39}$$

Then, by Theorem (4.10) we know that $\mathbb{P}$-a.s. for all $x \in B_{d_\ell}(z)$, $\pi_\ell^{1,\delta_z}(x) \le \widetilde{\pi}_\ell^1(z-x)$. Let us now control the dependence of $\mathcal{P}_y(B_1)$ on the random environment. By (3.9) of Lemma 3.2 in [40] we have for all $x \in \bigcup_{z\in\overline{\mathcal{A}}_\ell} B_{d_\ell}(z)$,

$$\mathcal{P}_x(B_1) \le e^{-c_4(\log c_\ell)^2} + \textstyle\sum_{y\in B_{\overline{\theta}_\ell}(x)} \mathbb{1}_{y\in\mathcal{T}_\ell} \sum_{y'\in B_{\overline{\theta}_\ell}(x)} \mathbb{1}_{y'\in\mathcal{T}_\ell} \equiv p_1(x) + p_2(x). \tag{4.2.40}$$

The contribution coming from $p_1(x)$ to (4.2.38) tends to zero. Thus, it suffices to control the contribution coming from $p_2(x)$ to (4.2.38). We bound $\mu(z) \le \mu_1(z) + \mu_2(z)$ and control the contribution coming from μ_i and p_2 to (4.2.38) separately. We call (I') the contribution coming from μ_1 and p_2, and likewise (II') that coming from μ_2 and p_2. Since, μ_2 is independent of p_2, $\mathbb{E}(II')$ is bounded above by

$$k_\ell(1)\mathbb{E}(p_2(0)) \textstyle\sum_z \mathbb{E}(\mu_2(z)) \sum_x \widetilde{\pi}_\ell^1(z-x) \le (\mathbb{1}_{d=2} + \mathbb{1}_{d\ge3}(\log\theta_\ell)^2)\mathbb{E}(p_2(0)), \tag{4.2.41}$$

where we used that μ_2 is a probability measure and the definition of $\widetilde{\pi}$. Now, $\mathbb{E}(p_2(0)) \le c_\ell^{-\alpha}\epsilon_\ell^{2-\alpha}$, and so (4.2.41) vanishes as $\ell \to \infty$. It remains to bound (I'). We bound $\mu_1(z) \le \mathbb{1}_{z\in\mathcal{T}_\ell}$. Using the definition of $\widetilde{\pi}$, one can show that

$$k_\ell(1) \textstyle\sum_{x\in B_{\overline{\theta}_\ell}(z)} \widetilde{\pi}_\ell^1(z-x) \le \overline{\theta}_\ell^2\theta_\ell^{-1}, \tag{4.2.42}$$

and so (I') is bounded above by $c_\ell^{-\alpha}\overline{\theta}_\ell^2\epsilon_\ell\theta_\ell^{-1}|\overline{\mathcal{A}}_\ell|$, which tends to zero by (4.1.14) and Condition (A-2). The proof of (4.2.38) is complete. It remains to establish that (II) vanishes $\mathbb{P}$-a.s. As in (4.2.40), we know by (3.8) of Lemma 3.2 in [40] that the contribution to $\mathcal{E}_x\overline{Z}_{\ell,1}\mathbb{1}_{\overline{Z}_{\ell,1}\le\varepsilon}$ coming from $y \notin B_{\overline{\theta}_\ell}(x)$ vanishes $\mathbb{P}$-a.s. Also, by (4.2.38), $\mathbb{P}$-a.s., $\mathbb{1}_{\overline{Z}_{\ell,1}\le\varepsilon} = \mathbb{1}_{\tau(y)\ell_{\theta_\ell}(y)\le c_\ell\varepsilon}$, and hence (II) tends to zero. Thus, (C-2')-(C-4') imply the conditions of Theorem 1.1 in [40]. The proof of Proposition 4.9 is done.

4.2.4 Verification of (C-2')-(C-4')

Let us establish that the assumptions of Proposition 4.9 are verified. Let $u > 0$, $t > 0$, and $\varepsilon > 0$ be fixed. Below, we state three lemmata that are sufficient to conclude that Condition (C-2')-(C-4') are verified. Rather than presenting their proofs, we first complete the verification of the assumptions of Proposition 4.9, supposing their claims. The proofs of these lemmata are postponed to the end of this section.

We begin with the verification of (C-2'). Due to the similarity between (C-2') and (C-2) of Proposition 3.6 in [40], let us recall the idea of proof in [40]. First it is established (see Lemma 4.1 in [40]) that

$$\lim_{\ell\to\infty} k_\ell(t) \sum_{y\in\mathbb{Z}^d} \mathbb{E}[Q_\ell^u(0,y)] = t\mathcal{K}u^{-\alpha}. \tag{4.2.43}$$

Since the key quantity in (C-2) is $\nu_\ell^t(u,\infty) = k_\ell(t)\sum_x\sum_y \mathbb{E}[\pi_\ell^{t,\delta_0(x)}]Q_\ell^u(x,y)$ it then remains to prove that $\mathbb{P}$-a.s. $\nu_\ell^t(u,\infty)$ concentrates around its mean. The extra difficulty in (C-2') is that $\widetilde{\nu}_\ell^{t,\mu}$ is defined with the random measure π_ℓ^t. Since we only have control on the convergence speed of $|Q_\ell^u(x,y) - \mathbb{E}Q_\ell^u(x,y)|$, but not on that of $|\pi_\ell^t - \mathbb{E}\pi_\ell^t|$, we set

$$\bar{\nu}_\ell^{t,\mu}(u,\infty) \equiv k_\ell(t)\sum_{x\in B_{d_\ell(t)}} \pi_\ell^{t,\mu}(x)\sum_{y\in\mathbb{Z}^d}\mathbb{E}[Q_\ell^u(x,y)], \tag{4.2.44}$$

and bound,

$$\mathbb{P}\big(|\widetilde{\nu}_\ell^{\mu,t}(u,\infty) - t\mathcal{K}u^{-\alpha}| > \varepsilon\big) \le \mathbb{P}\big(|\widetilde{\nu}_\ell^{\mu,t}(u,\infty) - \bar{\nu}_\ell^{t,\mu}(u,\infty)| > \varepsilon/3\big) \tag{4.2.45}$$

$$+\ \mathbb{P}\big(|\bar{\nu}_\ell^{t,\mu}(u,\infty) - \mathbb{E}\bar{\nu}_\ell^{\mu,t}(u,\infty)| > \varepsilon/3\big) \tag{4.2.46}$$

$$+\ \mathbb{1}_{|\mathbb{E}\bar{\nu}_\ell^{t,\mu}(u,\infty) - t\mathcal{K}u^{-\alpha}|>\varepsilon/3}. \tag{4.2.47}$$

Now, by (4.2.43), (4.2.47) is non-zero only for finitely many ℓ. Thus, it remains to show that, $\mathbb{P}$-a.s., (4.2.45) and (4.2.46) tend to zero. To bound (4.2.45), we distinguish whether $z \in \mathcal{T}_\ell$ or $z \in \mathcal{B}_\ell$ and bound (4.2.45) from above by

$$|\overline{\mathcal{A}}_\ell|\mathbb{P}\big(\mathbb{1}_{0\in\mathcal{T}_\ell}|\widetilde{\nu}_\ell^{\delta_0,t}(u,\infty) - \bar{\nu}_\ell^{\delta_0,t}(u,\infty)| > \tfrac{\varepsilon}{6}\big) + \mathbb{P}\big(|\widetilde{\nu}_\ell^{\mu_2,t}(u,\infty) - \bar{\nu}_\ell^{\mu_2,t}(u,\infty)| > \tfrac{\varepsilon}{6}\big), \tag{4.2.48}$$

where $\mu_2(z) = \mu(z|\overline{\mathcal{A}}_\ell \cap \mathcal{B}_\ell)$. The following two lemmata control (4.2.48) and (4.2.46).

Lemma 4.11. *There exists $\kappa \equiv \kappa(d) \in (0,\infty)$ and there exists $K(u,t) \in (0,\infty)$, increasing in t and decreasing in u, such that the following holds true for all $u > 0$ and $t > 0$ for n large enough. For $\mu' \in \{\mu_2, \delta_0\}$, setting $Y(\mu') \equiv \mathbb{1}_{0\in\mathcal{T}_\ell}\mathbb{1}_{\mu'=\delta_0} + \mathbb{1}_{\mu'=\mu_2}$, we have that*

$$\mathbb{P}(Y(\mu')|\widetilde{\nu}_\ell^{t,\mu'}(u,\infty) - \bar{\nu}_\ell^t(u,\infty)| > \varepsilon/3) \le \mathbb{P}(Y(\mu') = 1)\rho_\ell^{u,t}. \tag{4.2.49}$$

Lemma 4.12. *For all $u > 0$, $t > 0$, for ℓ large enough*

$$\mathbb{P}(|\bar{\nu}_\ell^t(u,\infty) - \mathbb{E}\bar{\nu}_\ell^t(u,\infty)| > \varepsilon/3) \ll \rho_\ell^{u,t}, \tag{4.2.50}$$

where $\rho_\ell^{u,t} \equiv \frac{K(u,t)}{\varepsilon^2}\big(\frac{(\log\theta_\ell)^\kappa}{\sqrt{\theta_\ell}}\mathbb{1}_{d\ge3} + \frac{(\log\log a_\ell)^\kappa}{\log a_\ell}\mathbb{1}_{d=2}\big)$.

Now, supposing Lemmata 4.11 and 4.12, (4.2.48) and (4.2.46) are bounded above by $\rho_\ell^{u,t}(|\overline{\mathcal{A}}_\ell|+1)$, which by (4.2.16) is smaller than n^{-2}. Thus, by Borel-Cantelli Lemma we conclude that, $\mathbb{P}$-a.s. $|\widetilde{\nu}_\ell^{\mu,t}(u,\infty) - t\mathcal{K}u^{-\alpha}|$ tends to zero as $\ell\to\infty$. This finishes the verification of (C-2'). Proceeding in the same way, and using in (4.2.43) the results from Section 4.4 in [40] (instead of Lemma 4.1 in [40]), one can show that the same is true for (C-4'). It remains to verify (C-3'). By a first order Chebyshev inequality, and proceeding as in (4.2.48),

$$\mathbb{P}(\widetilde{\sigma}_\ell^{\mu,t}(u,\infty) > \varepsilon) \le \frac{|\overline{\mathcal{A}}_\ell|k_\ell(t)}{\varepsilon}\sum_{x\in B_{d_\ell(t)}}\mathbb{E}\big[Y(\delta_0)\pi_\ell^{t,\delta_0}(x)\big(\textstyle\sum_{y\in\mathbb{Z}^d}Q_\ell^u(x,y)\big)^2\big]$$

$$+\ \frac{k_\ell(t)}{\varepsilon}\sum_{x\in B_{d_\ell(t)}(\overline{\mathcal{A}}_\ell)}\mathbb{E}\big[Y(\mu_2)\pi_\ell^{t,\mu_2}(x)\big(\textstyle\sum_{y\in\mathbb{Z}^d}Q_\ell^u(x,y)\big)^2\big]. \tag{4.2.51}$$

The following lemma bounds (4.2.51).

Lemma 4.13. *For all $u > 0$, $t > 0$, for ℓ large enough, we have for $\mu' \in \{\mu_2, \delta_0\}$,*

$$k_\ell(t)\mathbb{E}\big[\textstyle\sum_{x\in B_{d_\ell(t)}(\overline{\mathcal{A}}_\ell)} Y(\mu')\pi_\ell^{t,\mu'}(x)\big(\sum_{y\in\mathbb{Z}^d} Q_\ell^u(x,y)\big)^2\big] \le \mathbb{P}(Y(\mu')=1)\rho_\ell^{u,t}. \tag{4.2.52}$$

Supposing Lemma 4.13, (4.2.51) is bounded above by n^{-2}. By Borel-Cantelli Lemma this implies that, $\mathbb{P}$-a.s. $\widetilde{\sigma}_\ell^{\mu,t}(u,\infty)$ tends to zero as $\ell \to \infty$. Thus, (C-3') is satisfied.

The verification of the conditions of Proposition 4.9 is complete. Hence the proof that $\mathbb{P}$-a.s $\overline{S}_\ell^b \Rightarrow V_\alpha$, is finished.

It remains to establish the claims of Lemmata 4.11-4.13. We prove them in the following order. We begin with Lemma 4.11 in Section 4.2.4. We then we present the proof of Lemma 4.13 in Section 4.2.4. Finally, Section 4.2.4 contains the proof of Lemma 4.12. One important ingredient to their proofs is the following lemma. To shorten notation, set

$$Q_\ell^u(x) \equiv \textstyle\sum_{y\in\mathbb{Z}^d} Q_\ell^u(x,y), \qquad \overline{Q}_\ell^u(x) \equiv Q_\ell^u(x) - \mathbb{E}Q_\ell^u(x), \qquad x \in \mathbb{Z}^d. \tag{4.2.53}$$

Lemma 4.14. *For all $u > 0$, for ℓ large enough*

$$\mathbb{E}(Q_\ell^u(0))^2 \le \textstyle\sum_{y\in B_{\overline{\theta}_\ell}} \mathbb{E}(Q_\ell^u(0,y))^2 + (c_\ell\epsilon_\ell)^{-2\alpha}|B_{\overline{\theta}_\ell}|^2 \le \rho_\ell^u, \tag{4.2.54}$$

where $\rho_\ell^u \equiv K(u)a_\ell^{-1}(\sqrt{\theta_\ell}\mathbb{1}_{d\ge3} + \theta_\ell/\log\theta_\ell\mathbb{1}_{d=2})$.

Proof. The term $(c_\ell\epsilon_\ell)^{-2\alpha}|B_{\overline{\theta}_\ell}|^2$ bounds contribution coming from pairs $y', y \in B_{\overline{\theta}_\ell} \cap \mathcal{T}_\ell$. By construction of θ_ℓ it is of smaller order than ρ_ℓ^u. Thus, it suffices to show that the sum in (4.2.54) is bounded above by ρ_ℓ^u. This follows from Lemma 4.3 in [40]. □

Proof of Lemma 4.11

By a second order Markov inequality, the probability in (4.2.49) is for $\mu' \in \{\mu_2, \delta_0\}$ bounded from above by

$$\begin{aligned} &k_\ell^2(t)\mathbb{E}\big[Y(\mu')\textstyle\sum_{z\in\overline{\mathcal{A}}_\ell^{\mu'}} \mu'(z)\sum_{x\in B_{d_\ell(t)}(z)\setminus B_{\overline{\theta}_\ell}(z)} \pi_\ell^{t,\delta_z}(x)Q_\ell^u(x)\big]^2 \\ &+ k_\ell^2(t)\mathbb{E}\big[Y(\mu')\textstyle\sum_{x,x'\in B_{d_\ell(t)}(\overline{\mathcal{A}}_\ell^{\mu'}):x\ne x'} \pi_\ell^{t,\mu'}(x)\pi_\ell^{t,\mu'}(x')\overline{Q}_\ell^u(x)\overline{Q}_\ell^u(x')\big] \equiv (I)+(II), \end{aligned} \tag{4.2.55}$$

where for a set B we write $B_{d_\ell(t)}(B) \equiv \{x : \exists z \in B : x \in B_{d_\ell(t)}(z)\}$ and where $A^{\mu'}$ denotes the support of μ'. In order to bound (I) and (II) we must control the dependence between π_ℓ, Q_ℓ, and Y. For this, we use the Hölder continuity (see Proposition 3.2 in [5]) of $P_\mu(J(t)=x)$ and approximate $P_\mu(J(t)=x)$ by $P_\mu(J(t)=x, \eta(B)>t)$, where $\eta(B)$ is the exit time of the ball B, and B is a carefully chosen set (see Step 5). Due to the size of B, this method of proof only works for $d \ge 3$, and hence we assume in dimension $d=2$ that J is not random in the environment.

We begin with $d=2$. We distinguish whether $\mu' = \delta_0$ or $\mu' = \mu_2$. Let first $\mu' = \delta_0$. Then, since $\overline{Q}_\ell^u(x)$ depends only on $\tau(y)\mathbb{1}_{y\in\mathcal{T}_\ell}$ for $y \in B_{\overline{\theta}_\ell}(x)$, the $\overline{Q}_\ell^u$'s in (4.2.55) are independent of $Y(\delta_0)$. Thus, using $\mathbb{P}(Y(\mu')=1) = \mathbb{P}(0\in\mathcal{T}_\ell)$, we get that

$$\begin{aligned} (I)+(II) \le\; &\mathbb{P}(0\in\mathcal{T}_\ell)k_\ell^2(t)\textstyle\sum_{x\in B_{d_\ell(t)}}(\pi_\ell^{t,\delta_0}(x))^2\mathbb{E}\big(Q_\ell^u(x)\big)^2 \\ &+ \mathbb{P}(0\in\mathcal{T}_\ell)k_\ell^2(t)\textstyle\sum_{x\ne x'}\pi_\ell^{t,\delta_0}(x)\pi_\ell^{t,\delta_0}(x')\mathbb{E}\big[\overline{Q}_\ell^u(x)\overline{Q}_\ell^u(x')\big]\}. \end{aligned} \tag{4.2.56}$$

By Lemma 4.2 in [40] this is smaller than $\mathbb{P}(0\in\mathcal{T}_\ell)K(t)a_\ell\theta_\ell^{-1}\rho_\ell^u \le \rho_\ell^{u,t}$, as desired. Now let $\mu' = \mu_2$. In order to bound (4.2.55), we note that, since μ_2 only depends on $y \in \overline{\mathcal{A}}_\ell \cap \mathcal{B}_\ell$, π_ℓ^{t,μ_2} is independent of the $\overline{Q}_\ell^u$'s as well. Hence,

$$\begin{aligned} (I)+(II) \le\; &k_\ell^2(t)\textstyle\sum_{x\in B_{d_\ell(t)}(\overline{\mathcal{A}}_\ell^{\mu_2})}\mathbb{E}\big[(\pi_\ell^{t,\mu_2}(x))^2\mathbb{E}\big(Q_\ell^u(x)\big)^2\big] \\ &+ k_\ell^2(t)\textstyle\sum_{x,x'\in B_{d_\ell(t)}(\overline{\mathcal{A}}_\ell^{\mu_2}):x\ne x'}\mathbb{E}\big[\pi_\ell^{t,\mu_2}(x)\pi_\ell^{t,\mu_2}(x')\mathbb{E}\big[\overline{Q}_\ell^u(x)\overline{Q}_\ell^u(x')\big]\}\big]. \end{aligned} \tag{4.2.57}$$

Bounding $\pi_\ell^{t,\mu_2}(x) \leq \theta_\ell^{-1} \log a_\ell$, we get by Lemma 4.2 in [40] that the first summand in (4.2.57) is smaller than $K(t)\rho_\ell^u \theta_\ell^{-1} \log a_\ell \leq c_\ell^\alpha \epsilon_\ell^\alpha \rho_\ell^{u,t}$, as desired. Let us now control the second summand in (4.2.57). By independence of the $\overline{Q}_\ell^u$'s, the sum is only over x, x' for which $|x-x'| \leq 2\overline{\theta}_\ell$. We distinguish whether the distance $|x-x'|$ is smaller or larger than $\theta_\ell^{1/2} \log\log c_\ell$. When it is smaller, we bound $\mathbb{E}[\overline{Q}_\ell^u(x)\overline{Q}_\ell^u(x')] \leq \rho_\ell^u$, and when it is larger we use (3.8) of Lemma 3.2 in [40] to bound $\mathbb{E}[\overline{Q}_\ell^u(x)\overline{Q}_\ell^u(x')] \leq \theta_\ell c_\ell^{-\alpha} e^{-c_4(\log\log c_\ell)^2}$. Now, using (4.2.14) of Theorem 4.10 one can show that, for every $x \in B_{d_\ell(t)}\overline{\mathcal{A}}_\ell^{\mu_2}$, every $r \geq \theta_\ell^{1/2}$,

$$\textstyle\sum_{x':|x-x'|\leq r} k_\ell(t)\pi_\ell^{t,\mu'}(x') \leq K(t)r^2/\theta_\ell, \tag{4.2.58}$$

where $K(t) \in (0,\infty)$. Thus, the second summand in (4.2.57) is bounded above by

$$K(t)k_\ell(t)\{\rho_\ell^u(\log\log c_\ell)^2 + (\log c_\ell)^2 c_\ell^{-\alpha} e^{-c_4(\log\log c_\ell)^2}\}, \tag{4.2.59}$$

which is as claimed in (4.2.49). The proof of Lemma 4.11 is finished for $d=2$.

Let $d \geq 3$. The remainder of the proof comes in five steps, which we now briefly explain. First, we bound the contribution to (4.2.55) coming from x, x' that are such that $\{x,x'\} \cap B_{2\overline{\theta}_\ell}(\overline{\mathcal{A}}_\ell^{\mu_2}) \neq \emptyset$. This is needed in order to get that μ_2 and the Q_ℓ^u's are independent. Then, we bound the contribution to (I) coming from $x \in B_{d_\ell(t)}^{\mu'} \equiv B_{d_\ell(t)}(\overline{\mathcal{A}}_\ell^{\mu'}) \setminus B_{\overline{\theta}_\ell}(\overline{\mathcal{A}}_\ell^{\mu'})$. The remaining steps are devoted to the control the contribution to (II) coming from $(x,x') \in B_{d_\ell(t)}^{\mu'} \times B_{d_\ell(t)}^{\mu'}$. For this, we divide the set $B_{d_\ell(t)}^{\mu'} \times B_{d_\ell(t)}^{\mu'}$ into three subsets $B_1^{\mu'}, \ldots, B_3^{\mu'}$ as follows. Writing $|x - \overline{\mathcal{A}}_\ell^{\mu'}| \equiv \min_{y\in\overline{\mathcal{A}}_\ell^{\mu'}} |x-y|$, we set

$$B_1^{\mu'} \equiv \{(x,x') \in (B_{d_\ell(t)}^{\mu'})^2 : 0 < |x-x'| \leq 2\overline{\theta}_\ell\}, \tag{4.2.60}$$

$$B_2^{\mu'} \equiv \{(x,x') \in (B_{d_\ell(t)}^{\mu'})^2 \setminus B_1^{\mu'} : |x - \overline{\mathcal{A}}_\ell^{\mu'}| \wedge |x' - \overline{\mathcal{A}}_\ell^{\mu'}| \wedge |x-x'| \leq a_\ell^{1/2}\theta_\ell^{-1/4}\}, \tag{4.2.61}$$

$$B_3^{\mu'} \equiv \{(x,x') \in (B_{d_\ell(t)}^{\mu'})^2 \setminus B_1^{\mu'} : |x - \overline{\mathcal{A}}_\ell^{\mu'}| \wedge |x' - \overline{\mathcal{A}}_\ell^{\mu'}| \wedge |x-x'| > a_\ell^{1/2}\theta_\ell^{-1/4}\}. \tag{4.2.62}$$

Step 1. Let us now establish that the contribution to (4.2.55) coming from $x, x' : \{x,x'\} \cap B_{2\overline{\theta}_\ell}(\overline{\mathcal{A}}_\ell^{\mu_2}) \neq \emptyset$ is smaller than $(c_\ell\epsilon_\ell)^\alpha \rho_\ell^{u,t}$. For this, we construct a $\mathbb{P}$-a.s. bound on $\overline{Q}_\ell^u(x)$ for all $x \in B_{d_\ell(t)}(\overline{\mathcal{A}}_\ell^{\mu_2})$, which is independent of μ_2. For all $x \in B_{d_\ell(t)}(\overline{\mathcal{A}}_\ell^{\mu_2})$ we have that

$$Q_\ell^u(x) \leq \textstyle\sum_{z\in B_{\overline{\theta}_\ell}} \mathbb{1}_{z\in\mathcal{T}_\ell} P_x(\sigma(z) \leq \theta_\ell) \leq \sum_{z\in B_{\overline{\theta}_\ell}} \mathbb{1}_{z\in\mathcal{T}_\ell}\big(\frac{|x-z|}{(\log a_\ell)^4}\big)^{2-d} \equiv q_\ell^u(y), \tag{4.2.63}$$

where we used that $P_x(\sigma(z) \leq \theta_\ell) \leq P_x(\sigma(z) < \infty) \leq c|x-z|^{2-d}U_z^{d-2}$ and that $\mathbb{P}$-a.s., $U_z \ll (\log a_\ell)^4$ for all $z \in B_{d_\ell(t)}$. We are ready to control the contribution to (I) coming from $x, x' : \{x,x'\} \cap B_{2\overline{\theta}_\ell}(\overline{\mathcal{A}}_\ell^{\mu_2}) \neq \emptyset$. We have that,

$$\begin{aligned}\textstyle\sum_{x\in B_{2\overline{\theta}_\ell}(\overline{\mathcal{A}}_\ell)} \mathbb{E}[\pi_\ell^{\mu_2,t}(x)Q_\ell^u(x)] &\leq \theta_\ell^{-1}\textstyle\sum_{x\in B_{2\overline{\theta}_\ell}(\overline{\mathcal{A}}_\ell^{\mu_2})}\sum_{z\in\overline{\mathcal{A}}_\ell^{\mu_2}} \mathbb{E}[\mu_2(z)\widetilde{\pi}_\ell^t(z-x)q_\ell^u(x)] \\ &= \theta_\ell^{-1}\textstyle\sum_{x\in B_{2\overline{\theta}_\ell}} \widetilde{\pi}_\ell^t(x)\mathbb{E}[q_\ell^u(x)]. \end{aligned}\tag{4.2.64}$$

One can check that $\mathbb{E}[q_\ell^u(x)] \leq (c_\ell\epsilon_\ell)^{-\alpha}(\log a_\ell)^{4d}$. Also,

$$\textstyle\sum_{z\in\overline{\mathcal{A}}_\ell^{\mu_2}}\sum_{x\in B_{2\overline{\theta}_\ell}(\overline{\mathcal{A}}_\ell^{\mu_2})} \mu_2(z)\widetilde{\pi}_\ell^t(z-x) \leq \overline{\theta}_\ell^{d-1}\sum_{z\in\overline{\mathcal{A}}_\ell^{\mu_2}} \mu_2(z)|z-x|^{d-2} \leq \overline{\theta}_\ell^d a_\ell^{2/d}, \tag{4.2.65}$$

where we used that $|\overline{\mathcal{A}}_\ell^{\mu_2}| = |B_{a_\ell^{1/d}}|$ and the fact that $|z-x|^{d-2}$ is decreasing in $|z-x|$. Thus, the contribution coming from $x \in B_{2\overline{\theta}_\ell}(\overline{\mathcal{A}}_\ell^{\mu_2})$ to (I) is at most $c_\ell^{-2\alpha/3}(\log c_\ell)^\kappa$, for $\kappa \in (0,\infty)$.

This is smaller than the right hand side of (4.2.49). In order to bound the contribution coming from $x, x' : \{x, x'\} \cap B_{2\overline{\theta}_\ell}(\overline{\mathcal{A}}_\ell^{\mu_2}) \neq \emptyset$ to (II), we bound $\pi_\ell^{t,\mu_2}(y) \leq \sum_z \mu_2(z)\tilde{\pi}_\ell^t(z-y)$ and $\overline{Q}_\ell^u(y) \leq q_\ell^u(y)$ for $y \in \{x, x'\}$. Then, we note that for x, x' such that $|x - x'| > 2\overline{\theta}_\ell$ the q_ℓ^u's are independent of each other and of μ_2 and we get from (4.2.65) that their contribution to (II) is bounded above by $(c_\ell \epsilon_\ell)^{-2\alpha}(\log a_\ell)^{8d}\overline{\theta}_\ell^{2d} a_\ell^{4/d}$ which is smaller than the right hand side of (4.2.49). When x, x' are such that $|x - x'| \leq 2\overline{\theta}_\ell$, we bound by Cauchy-Schwarz inequality, $\mathbb{E}q_\ell^u(x)q_\ell^u(x') \leq \mathbb{E}q_\ell^u(0)$. Then, we use that,

$$\textstyle\sum_{x':|x-x'|\leq 2\overline{\theta}_\ell} \mu_2(z)\tilde{\pi}_\ell^t(z-y) \leq (\log c_\ell)^2, \quad \forall x, z, \tag{4.2.66}$$

and get, together with (4.2.65), that the contribution coming from $x \in B_{2\overline{\theta}_\ell}(\overline{\mathcal{A}}_\ell^{\mu_2})$ and $x' \in B_{2\overline{\theta}_\ell}(x)$ is bounded above by $c_\ell^{-3\alpha/\alpha}(\log c_\ell)^{\kappa'}$, for $\kappa' \in (0, \infty)$. The first step is finished.

Step 2. In order to control (I), we first bound $\pi_\ell^{t,\mu'}(x) \leq \sum_z \mu'(z)\tilde{\pi}_\ell^t(x-z) \equiv \tilde{\pi}_\ell^{t,\mu'}(x)$ for $\mu' \in \{\delta_0, \mu_2\}$. Since by construction, Q_ℓ^u is independent of $Y(\mu')$ and μ_2, we get by Lemma 4.14 and Step 1 that

$$(I) \leq \rho_\ell^{u,t} + \mathbb{P}(Y(\mu') = 1) \textstyle\sum_{x \in B_{d_\ell(t)}, |x| > 2\theta_\ell} (k_\ell(t)\tilde{\pi}_\ell^t(x))^2 \rho_\ell^u \ll \rho_\ell^{u,t}, \tag{4.2.67}$$

as desired in (4.2.49).

Step 3. Let $(x, x') \in B_1^{\mu'}$. As in Step 2, we bound $\pi_\ell^{t,\mu'}(x) \leq \tilde{\pi}_\ell^{t,\mu'}(x)$ and get by independence of $Y(\mu')$, $Q_\ell^u(x)Q_\ell^u(x')$, and μ' that each summand in (II) is bounded above by

$$\mathbb{P}(Y(\mu') = 1)\mathbb{E}(\tilde{\pi}_\ell^{t,\mu'}(x)\tilde{\pi}_\ell^{t,\mu'}(x'))\big(\mathbb{E}Q_\ell^u(x)Q_\ell^u(x') + \big(\mathbb{E}Q_\ell^u(0)\big)^2\big). \tag{4.2.68}$$

By (4.2.39) there exists $C \in (0, \infty)$ such that

$$\textstyle\sum_{(x,x') \in B_1^{\mu'}} (k_\ell(t))^2 \mathbb{E}(\tilde{\pi}_\ell^{t,\mu'}(x)\tilde{\pi}_\ell^{t,\mu'}(x')) \leq (k_\ell(t))^2 \sum_{x \in B_{d_\ell(t)}, x' \in B_{2\theta_\ell}} C\tilde{\pi}_\ell^t(x)\tilde{\pi}_\ell^t(x'), \tag{4.2.69}$$

which is bounded above by $k_\ell(t)(\log c_\ell)^2$. Thus, the contribution coming from the second summand in (4.2.68) to (II) is bounded above by $c_\ell^{-3\alpha/2}$, as desired. It remains to bound the contribution coming from the first summand in (4.2.68). As in the proof of Lemma 4.2 in [40] (see (4.74)) one can show that

$$\textstyle\sum_{(x,x') \in B_1^{\mu'}} \mathbb{E}(\tilde{\pi}_\ell^{t,\mu'}(x)\tilde{\pi}_\ell^{t,\mu'}(x'))\mathbb{E}Q_\ell^u(x)Q_\ell^u(x') \leq K(u,t)(\log \theta_\ell)^2 \theta_\ell^{-1/2}, \tag{4.2.70}$$

which is bounded above by $\rho_\ell^{u,t}$. This finishes the third step.

Step 4. Let $(x, x') \in B_2^{\mu'}$. Then, $Q_\ell^u(x)$ and $Q_\ell^u(x')$ are independent of each other, $Y(\mu')$, and of μ'. Thus,

$$\begin{aligned} &\mathbb{E}[Y(\mu')\pi_\ell^{t,\mu'}(x)\pi_\ell^{t,\mu'}(x')\overline{Q}_\ell^u(x)\overline{Q}_\ell^u(x')] \leq 4\mathbb{E}[\tilde{\pi}_\ell^{t,\mu'}(x)\tilde{\pi}_\ell^{t,\mu'}(x')]\mathbb{P}(Y(\mu') = 1)(\mathbb{E}Q_\ell^u(0))^2 \\ &\leq K''(u)\theta_\ell^2 c_\ell^{-2\alpha}\mathbb{P}(Y(\mu') = 1)\mathbb{E}[\tilde{\pi}_\ell^{t,\mu'}(x)\tilde{\pi}_\ell^{t,\mu'}(x')], \end{aligned} \tag{4.2.71}$$

where we used (4.4) in [40] to bound $(\mathbb{E}Q_\ell^u(0))^2 \leq K''(u)\theta_\ell c_\ell^{-\alpha}$ for $K'' \in (0, \infty)$. By (4.2.39), the construction of $B_2^{\mu'}$ and as in (4.2.70),

$$\begin{aligned} &(k_\ell(t))^2 \textstyle\sum_{(x,x') \in B_2^{\mu'}} \mathbb{E}[\tilde{\pi}_\ell^{t,\mu'}(x)\tilde{\pi}_\ell^{t,\mu'}(x')] \\ &\leq C(k_\ell(t))^2 \textstyle\sum_{x \in B_{d_\ell(t)}} \tilde{\pi}_\ell^t(x) \sum_{|x'| \leq a_\ell^{1/2}\theta_\ell^{-1/4}} \tilde{\pi}_\ell^t(x') \leq 2k_\ell(t)a_\ell\theta_\ell^{-3/2}. \end{aligned} \tag{4.2.72}$$

Putting (4.2.71) and (4.2.72) together, we see that the contribution to (II) coming from $(x, x') \in B_2^{\mu'}$ satisfies (4.2.49).

Step 5. Finally, let $(x,x') \in B_3^{\mu'}$. We construct an approximation of $\pi_\ell^{\mu',t,*}$, denoted by $p_\ell^{\mu',t}$, which is independent of the Q_ℓ^u's. Then we use that

$$\mathbb{E}[p_\ell^{\mu',t}(x)p_\ell^{\mu',t}(x')\overline{Q}_\ell^u(x)\overline{Q}_\ell^u(x')Y(\mu')] = \mathbb{E}[p_\ell^{\mu',t}(x)p_\ell^{\mu',t}(x')Y(\mu')]\mathbb{E}(\overline{Q}_\ell^u(x)\overline{Q}_\ell^u(x')), \tag{4.2.73}$$

which is equal to zero. It remains to construct $p_\ell^{\mu',t}$ and control $|\pi_\ell^{\mu',t} - p_\ell^{\mu',t}|$. Let x^* be such that $|x-x^*| = a_\ell^{3/8}$. Set $B \equiv B_{\theta_\ell}(x) \cup B_{\theta_\ell}(x')$. For $s>0$ let the heat kernel of J killed on the exit of B^c be given by

$$q_s^B(x^*) \equiv P_{\mu'}(J(s)=x^*, \eta(B^c) \geq s). \tag{4.2.74}$$

Define $p_\ell^{\mu',t} \equiv (k_\ell(t))^{-1}\sum_{k=1}^{k_\ell(t)-1} q_{k\theta_\ell}^B(x^*)$. By construction, $p_\ell^{\mu',t}(x)$ is independent of $\overline{Q}_\ell^u(x)$ and $\overline{Q}_\ell^u(x')$. For $y \in \{x,x'\}$ we write

$$\pi_\ell^{\mu',t}(y) = p_\ell^{\mu',t}(y) + (\pi_\ell^{\mu',t}(y^*) - p_\ell^{\mu',t}(y)) + (\pi_\ell^{\mu',t}(y) - \pi_\ell^{\mu',t}(y^*)), \tag{4.2.75}$$

and get, since (4.2.73) is equal to zero,

$$\begin{aligned}
&(k_\ell(t))^2\mathbb{E}[Y(\mu')\pi_\ell^{\mu',t}(x)\pi_\ell^{\mu',t}(x')\overline{Q}_\ell^u(x)\overline{Q}_\ell^u(x')] \\
\leq\ &(k_\ell(t))^2\mathbb{E}[Y(\mu')|\overline{Q}_\ell^u(x)||\overline{Q}_\ell^u(x')|\{|\pi_\ell^t(x^*) - p_\ell^{\mu',t}(x)| + |\pi_\ell^{\mu',t}(x) - \pi_\ell^{\mu',t}(x^*)|\} \\
&\qquad \times \{\pi_\ell^{\mu',t,*}(x') + |\pi_\ell^{\mu',t}(x'^*) - p_\ell^{\mu',t}(x')| + |\pi_\ell^{\mu',t}(x') - \pi_\ell^{\mu',t}(x'^*)|\}].
\end{aligned} \tag{4.2.76}$$

Since $|x'^*| = |x'|(1+o(1))$, there exists $c' \in (0,\infty)$ such that, for all $z \in \overline{\mathcal{A}}_\ell^{\mu'}$, $\pi_\ell^{\delta_z,t}(x'^*) \leq c'\tilde{\pi}_\ell(z-x')$ and $p_\ell^{\delta_z,t}(x') \leq c'\tilde{\pi}_\ell^t(z-x')$, and hence the second line on the right hand side of (4.2.76) is bounded above by $C\tilde{\pi}_\ell^t(z-x')$ for $C=6c'$. Suppose we can establish that, $\mathbb{P}$-a.s., for all $z \in \overline{\mathcal{A}}_\ell^{\mu'}$,

$$k_\ell(t)(|\pi_\ell^{\delta_z,t}(x^*) - p_\ell^{\mu',t}(x)| + |\pi_\ell^{\delta_z,t}(x) - \pi_\ell^{\delta_z,t}(x)|) \leq \bar{\delta}_\ell^1(x-z) + \bar{\delta}_\ell^2(x-z), \tag{4.2.77}$$

for sequences $\bar{\delta}_\ell^i(x-z)$ satisfying $\sum_{x \in B_{d_\ell(t)}(\overline{\mathcal{A}}_\ell^{\mu'})} \bar{\delta}_\ell^i(x-z) \leq \theta_\ell^{-3/2}c_\ell^\alpha$. Then, using (4.2.76), the contribution to (4.2.55) coming from $(x,x') \in B_3^{\mu'}$ is bounded above by

$$\begin{aligned}
&k_\ell(t)\textstyle\sum_{(x,x') \in B_3^{\mu'}} \tilde{\pi}_\ell^t(x')(\bar{\delta}_\ell^1(x) + \bar{\delta}_\ell^2(x)\mathbb{E}[Y(\mu')|\overline{Q}_\ell^u(x)||\overline{Q}_\ell^u(x')|] \\
\leq\ &\mathbb{P}(Y(\mu')=1)K'(t,u)c_\ell^{-\alpha}\theta_\ell \textstyle\sum_{x \in B_{d_\ell(t)}}(\bar{\delta}_\ell^1(x) + \bar{\delta}_\ell^2(x))\sum_{x' \in B_{d_\ell(t)}} \tilde{\pi}_\ell^t(x'),
\end{aligned} \tag{4.2.78}$$

which is as in (4.2.49). It remains to construct the $\bar{\delta}_\ell^i$'s. Without loss of generality, we construct the $\bar{\delta}_\ell^i$'s for $z=0$. Let first $i=1$. For $k \geq 1$ we bound

$$\begin{aligned}
&|q_{k\theta_\ell}(x^*) - q_{k\theta_\ell}^B(x^*)| \leq \textstyle\sum_{z \in \partial B} P(\sigma(z) \leq k\theta_\ell, J(\eta(B))=z, J(k\theta_\ell)=x^*) \\
\leq\ &\textstyle\sum_{z \in \partial B}\Big\{\frac{(\log|z-x^*|)^d}{|z-x^*|^d}P(\sigma(z) \leq k\theta_\ell, J(\eta(B))=z) + e^{-c_4(\log|z-x^*|)^2}\Big\},
\end{aligned} \tag{4.2.79}$$

where we used (3.8) of Lemma 3.2 in [40] and (4.2.13) of Theorem 4.10. Since $|z-x^*| > a_\ell^{3/8} - \theta_\ell$ the first summand is larger than the second and it suffices to bound the first. As in (4.2.63), $P(\sigma(z) \leq k\theta_\ell) \leq (\frac{z}{(\log a_\ell)^4})^{2-d}$ and get that (4.2.79) is, $\mathbb{P}$-a.s., bounded above by

$$\max_{z \in \partial B}\Big\{\frac{(\log|z-x^*|)^d}{|z-x^*|^d}P(\sigma(z) \leq k\theta_\ell)\Big\} \leq (\log a_\ell)^{5d}\max_{z \in \partial B}|z-x^*|^{-d}|z|^{2-d}. \tag{4.2.80}$$

By construction of B, B_3^1, and x^*, the maximum in (4.2.80) is smaller than $Ca_\ell^{-3d/8}|x|^{2-d}$. Therefore, $\mathbb{P}$-a.s., for n large enough

$$k_\ell(t)|\pi_\ell^{\delta_0,t}(x^*) - \pi_\ell^{\delta_0,t,*}(x)| \leq C'c_\ell^\alpha a_\ell^{-3d/8}|x|^{2-d} \ll c_\ell^{\alpha(1-3d/8)}|x|^{2-d} \equiv \bar{\delta}_\ell^1(x). \tag{4.2.81}$$

The sum over $x \in B_{d_\ell(t)}$ of $\bar\delta^1_\ell(x)$ is smaller than $Cc_\ell^{7\alpha/8}$, as desired. Let us now construct $\bar\delta^2_\ell$. By (3.8) of Lemma 3.2 in [40], $|q_{k\theta_\ell}(x) - q_{k\theta_\ell}(x^*)| \le e^{-c_4(\log a_\ell)^2}$ for $k\theta_\ell \le |x|^2(\log a_\ell)^{-2}$ and hence it suffices to considers $k\theta_\ell > |x|^2(\log a_\ell)^{-2}$. For such k, we use the Hölder continuity of the heat kernel resulting from a parabolic Harnack inequality (hereafter PHI). Specifically, we know by Proposition 3.2 in [5] that there exists $\beta \in (0,1)$ such that for $t > 0$ and $x \in \mathbb{Z}^d$ for which a PHI holds,

$$|q_{t^2}(x) - q_{t^2}(z)| \le (|x-z|/t)^\beta \sup\nolimits_{(u,y)\in[3/4t^2,t^2]\times B_{1/2t}(x)} q_u(y), \quad \forall z \in B_{1/2t}(x). \tag{4.2.82}$$

By Theorem 3.7 in [4] we know that a PHI holds for $t > 0$ and $x \in \mathbb{Z}^d$ for which $t > U_x$. Since $\sqrt{k\theta_\ell} \gg U_x$, and since $x^* \in B_{1/2\sqrt{k\theta_\ell}}(x)$, we get by (4.2.82)

$$|q_{k\theta_\ell}(x) - q_{k\theta_\ell}(x^*)| \le \tfrac{|x-x^*|^\beta}{(k\theta_\ell)^{\beta/2}} \sup\nolimits_{(u,y)\in[3/4k\theta_\ell,k\theta_\ell]\times B_{1/2\sqrt{k\theta_\ell}}(x)} q_u(y). \tag{4.2.83}$$

By Theorem 4.10, $q_u(y) \le c_1(3/4k\theta_\ell)^{-d/2}$ for all $(u,y) \in [3/4k\theta_\ell, k\theta_\ell] \times B_{1/2\sqrt{k\theta_\ell}}(x)$. Since $k\theta_\ell > |x|^2(\log a_\ell)^{-2}$ and by construction of x^*,

$$\begin{aligned} k_\ell(t)|\pi_\ell^{\delta_0,t}(x^*) - \pi_\ell^{\delta_0,t,*}(x)| &\le \textstyle\sum_{k=1}^{k_\ell(1)-1} |q_{k\theta_\ell}(x) - q_{k\theta_\ell}(x^*)| \\ &\le ck_\ell(t)a_\ell^{-\beta/8}|x|^{-d-\beta}(\log a_\ell)^{2(d+\beta)} \equiv \bar\delta^2_\ell(x). \end{aligned} \tag{4.2.84}$$

The sum over x of $\bar\delta^2_\ell(x)$ is bounded above by $c_\ell a_\ell^{-\beta}$. Hence, the contribution to (4.2.55) coming from $(x,x') \in B_3$ satisfies (4.2.49). This finishes the proof of Lemma 4.11.

Proof of Lemma 4.13

Let $u > 0$, $t > 0$. Using the $\mathbb{P}$-a.s. bound $\pi_\ell^{\delta_z,t}(x) \le \tilde\pi^t_\ell(x-z)$, it remains to bound

$$\textstyle\sum_{z\in\overline{\mathcal{A}}_\ell} \mathbb{E}\big[\mu'(z)Y(\mu')\sum_{x\in B_{d_\ell(t)}(z)\setminus B_{\overline{\theta}_\ell}(z)} \big(Q^u_\ell(x)\big)^2\tilde\pi^t_\ell(x-z)\big]. \tag{4.2.85}$$

When either $d = 2$ or $d \ge 3$ and $\mu' = \delta_0$, then the Q^u_ℓ's are independent of μ' and $Y(\mu')$ and Lemma 4.14 implies together with the construction of $\tilde\pi^t_\ell$ that (4.2.85) is bounded above by $\mathbb{P}(Y(\mu') = 1)\rho^{u,t}_\ell$. When $d \ge 3$ and $\mu' = \mu_2$, the same reasoning applies to the contribution to (4.2.85) coming from $x \in B^{\mu_2}_{d_\ell(t)}$. From Step 1 in the proof of Lemma 4.11 we know that the contribution to (4.2.85) coming from $x \in B_{d_\ell(t)}(\overline{\mathcal{A}}^{\mu_2}_\ell) \setminus B^{\mu_2}_{d_\ell(t)}$ is smaller than $\mathbb{P}(Y(\mu') = 1)\rho^{u,t}_\ell$. The proof of Lemma 4.13 is complete.

Proof of Lemma 4.12

When $\theta = 0$, $\bar\nu^t_\ell(u,\infty) = \mathbb{E}\bar\nu^t_\ell(u,\infty)$, and so (4.2.50) holds in this case. Let $\theta > 0$, and so by assumption $d \ge 3$. By a second order Markov inequality the probability in (4.2.50) is bounded by $(I) + (II)$, where

$$\begin{aligned} (I) &\equiv \textstyle\sum_{x\in B_{d_\ell(t)}(\overline{\mathcal{A}}_\ell)} (k_\ell(t)\mathbb{E}Q^u_\ell(0))^2\mathbb{E}\big[(\pi^{\mu,t}_\ell(x) - \mathbb{E}\pi^{\mu,t}_\ell(x))\big]^2, \\ (II) &\equiv \textstyle\sum_{x'\neq x} (k_\ell(t)\mathbb{E}Q^u_\ell(0))^2\mathbb{E}\big[(\pi^{\mu,t}_\ell(x) - \mathbb{E}\pi^{\mu,t}_\ell(x))(\pi^{\mu,t}_\ell(x') - \mathbb{E}\pi^{\mu,t}_\ell(x'))\big]. \end{aligned} \tag{4.2.86}$$

Let us first control (I). By (4.4) in [40], $(k_\ell(t)\mathbb{E}Q^u_\ell(0))^2 \le K'(u,t)$ for some $K'(u,t) \in (0,\infty)$, and so, by (4.2.13) of Theorem 4.10,

$$(I) \le K'(u,t)\theta_\ell^{-1}\log a_\ell \textstyle\sum_{x\in B_{d_\ell(t)}(\overline{\mathcal{A}}_\ell)} \mathbb{E}(\pi^{\mu,t}_\ell(x)) \le K''(u,t)\theta_\ell^{-1}\log a_\ell, \tag{4.2.87}$$

where $K''(u,t) \in (0,\infty)$, as claimed in (4.2.50). To bound (II), fix $x \in B_{d_\ell(t)}(\overline{\mathcal{A}}_\ell)$. Since $\pi_\ell^{\mu,t}$ is a probability measure,

$$\begin{aligned}&\mathbb{E}\big[(\pi_\ell^{\mu,t}(x) - \mathbb{E}\pi_\ell^{\mu,t}(x)) \textstyle\sum_{x' \in B_{d_\ell(t)}(\overline{\mathcal{A}}_\ell), x' \neq x} (\pi_\ell^{\mu,t}(x') - \mathbb{E}\pi_\ell^{\mu,t}(x'))\big] \\ &\leq \mathbb{E}\big[|\pi_\ell^{\mu,t}(x) - \mathbb{E}\pi_\ell^{\mu,t}(x)||\pi_\ell^{\mu,t}(x) - \mathbb{E}\pi_\ell^{\mu,t}(x)|\big], \end{aligned} \tag{4.2.88}$$

and therefore

$$(II) \leq \textstyle\sum_{x \in B_{d_\ell(t)}(\overline{\mathcal{A}}_\ell)} (k_\ell(t)\mathbb{E}Q_\ell^u(0))^2 \mathbb{E}[(\pi_\ell^{\mu,t}(x) - \mathbb{E}\pi_\ell^{\mu,t}(x))^2], \tag{4.2.89}$$

which by (4.2.87) is as claimed in (4.2.50). The proof of Lemma 4.12 is finished.

4.3 Independence of σ and V_α

In this section we prove assertion (ii) of Theorem 4.1, that is we establish that, $\mathbb{P}$-a.s., σ, viewed as an element of $D[0,\infty)$, is independent of V_α. For this, we establish that, $\mathbb{P}$-a.s., $\overline{S}_\ell^b$ is asymptotically independent of σ_ℓ. By Daniell-Kolmogorov extension theorem it suffices to prove that for every $l \in \mathbb{N}$, for all $0 < t_1 < t_2 \cdots < t_l < \infty$, the distributions of $(\overline{S}_\ell^b(t_1), \dots, \overline{S}_\ell^b(t_l))$ and σ_ℓ are independent of each other. Independence follows by Corollary 14.1 in [43] if we can establish for all $0 < \lambda_1, \dots, \lambda_l, \lambda < \infty$ that, for all $\varepsilon > 0$ there exists ℓ large enough such that

$$\big|\mathcal{E}_\mu e^{-\lambda\sigma_\ell - \sum_{i=1}^l \lambda_i \overline{S}_\ell^b(t_i)} - \mathcal{E}_\mu e^{-\lambda\sigma_\ell} \mathcal{E}_\mu e^{-\sum_{i=1}^l \lambda_i \overline{S}_\ell^b(t_i)}\big| < \varepsilon. \tag{4.3.1}$$

In order to obtain $\mathbb{P}$-a.s. independence between $\overline{S}_\ell^b$ and σ_ℓ, we first prove that (4.3.1) holds true $\mathbb{P}$-a.s. for one choice of the λ_i's and t_i's and that the absolute value in (4.3.1) can be bounded by an increasing function of t_l and $\max_i \lambda_i$. This implies by Lemma 2.1 in [40] that the claim of (4.3.1) holds, $\mathbb{P}$-a.s., for all $l \in \mathbb{N}$, all t_i's, and all λ_i's.

Let $l \in \mathbb{N}$. Take $0 < t_1 < t_2 \cdots < t_l < \infty$ and $0 < \lambda_1, \dots, \lambda_l, \lambda < \infty$. Choose ℓ large enough so that $k_\ell(t_1) > 1$. Let us now rewrite the expectation on the left hand side of (4.3.1). For $j = 1, \dots l$, write $\widetilde{\lambda}_j = \sum_{i=\min\{m: j < k_\ell(t_m)\}}^l \lambda_i \in (0,\infty)$, and observe that

$$\textstyle\sum_{i=1}^l \lambda_i \overline{S}_\ell^b(t_i) = \sum_{i=1}^l \lambda_i \sum_{j=1}^{k_\ell(t_i)-1} \overline{Z}_{\ell,j} = \sum_{j=1}^{k_\ell(t_l)-1} \widetilde{\lambda}_j \overline{Z}_{\ell,j}. \tag{4.3.2}$$

Thus, we may substitute the sum of the $\overline{Z}_{\ell,j}$'s for the sum of the $\overline{S}_\ell^b$'s in (4.3.1). Now, let $\mathcal{F}_{\ell,j} = \sigma(J(s), s \leq j\theta_\ell)$ be the array of sigma-algebras generated by J. Notice that σ_ℓ and $\overline{Z}_{\ell,j}$ are $\mathcal{F}_{\ell,j+1}$ measurable for $j \geq 0$, and so we obtain by the law of total expectation

$$\mathcal{E}_\mu\big(e^{-\lambda\sigma_\ell - \sum_{j=1}^{k_\ell(t_l)-1} \widetilde{\lambda}_j \overline{Z}_{\ell,j}}\big) = \mathcal{E}_\mu\big(e^{-\lambda\sigma_\ell} \textstyle\prod_{j=1}^{k_\ell(t_l)-1} \mathcal{E}_\mu(e^{-\widetilde{\lambda}_j \overline{Z}_{\ell,j}} | \mathcal{F}_{\ell,j})\big), \tag{4.3.3}$$

and similarly,

$$\mathcal{E}_\mu\big(e^{-\sum_{j=1}^{k_\ell(t_l)-1} \widetilde{\lambda}_j \overline{Z}_{\ell,j}}\big) = \mathcal{E}_\mu\big(e^{-\widetilde{\lambda}_1 \overline{Z}_{\ell,1}} \textstyle\prod_{j=2}^{k_\ell(t_l)-1} \mathcal{E}_\mu(e^{-\widetilde{\lambda}_j \overline{Z}_{\ell,j}} | \mathcal{F}_{\ell,j})\big). \tag{4.3.4}$$

Using (4.3.3) and (4.3.4) in (4.3.1), we see that the left hand side of (4.3.1) is bounded above by

$$\mathcal{E}_\mu \mathcal{L}_\ell(1) + \mathcal{E}_\mu(\mathcal{L}_\ell(2)), \tag{4.3.5}$$

where for $i = 1, 2$ we set

$$\mathcal{L}_\ell(i) \equiv \big|\textstyle\prod_{j=i}^{k_\ell(t_l)-1} \mathcal{E}_\mu(e^{-\widetilde{\lambda}_j \overline{Z}_{\ell,j}} | \mathcal{F}_{\ell,j}) - \prod_{j=i}^{k_\ell(t_l)-1} \mathcal{E}_\mu e^{-\widetilde{\lambda}_j \overline{Z}_{\ell,j}}\big|. \tag{4.3.6}$$

Thus, to establish the claim of (4.3.1) we now prove that $\mathbb{P}$-a.s., for $\varepsilon > 0$, for ℓ large enough, for $i = 1, 2$ we have $\mathcal{E}_\mu \mathcal{L}_\ell(i) < \varepsilon$. Without loss of generality we present the proof for $i = 1$ only. For this, we rewrite $\mathcal{L}_\ell(1)$. Notice that by partial integration, the Laplace transform of a random variable $X \geq 0$ is given by

$$\mathcal{E} e^{-uX} = 1 - \int_0^\infty e^{-v} \mathcal{P}(Xu > v) dv, \quad u > 0. \tag{4.3.7}$$

Using (4.3.7) and $\prod_i (1 - y_i) \sim 1 - \sum_i y_i$ for small y_i,

$$\begin{aligned} \mathcal{L}_\ell(1) &\sim \Big|\sum_{j=1}^{k_\ell(t_l)-1} \int_0^\infty e^{-v} [\mathcal{P}_\mu(\widetilde{\lambda}_j \overline{Z}_{\ell,j} > v | \mathcal{F}_{\ell,j}) - \mathcal{P}_\mu(\widetilde{\lambda}_j \overline{Z}_{\ell,j} > v)] dv\Big| \\ &= \Big|\int_0^\infty e^{-v} [\nu_\ell^{J,t}(v, \widetilde{\lambda}) - \mathcal{E}_\mu \nu_\ell^{J,t}(v, \widetilde{\lambda})] dv\Big|, \end{aligned} \tag{4.3.8}$$

where for $v, t > 0$, and $\widetilde{\lambda} \equiv (\widetilde{\lambda}_j, i = j, \dots k_\ell(t) - 1)$ we write

$$\nu_\ell^{J,t}(v, \widetilde{\lambda}) \equiv \sum_{j=1}^{k_\ell(t)-1} \mathcal{P}_\mu(\widetilde{\lambda}_j \overline{Z}_{\ell,j} > v | \mathcal{F}_{\ell,j}). \tag{4.3.9}$$

We define

$$A_\ell^\varepsilon \equiv \{|\int_0^\infty e^{-v} [\nu_\ell^{J,t_l}(v, \widetilde{\lambda}) - \mathcal{E}_\mu \nu_\ell^{J,t_l}(v, \widetilde{\lambda})] dv| > \varepsilon\}. \tag{4.3.10}$$

Now, since $\mathcal{L}_\ell(1) \leq 1$,

$$\mathcal{E}_\mu [\mathcal{L}_\ell(1)(\mathbb{1}_{(\overline{\mathcal{A}}_\ell^\varepsilon)^c} + \mathbb{1}_{\overline{\mathcal{A}}_\ell^\varepsilon})] \leq \varepsilon + \mathcal{P}_\mu(\overline{\mathcal{A}}_\ell^\varepsilon(x)). \tag{4.3.11}$$

Writing $\overline{\nu}_\ell^{J,t}(v, \widetilde{\lambda}) = \nu_\ell^{J,t}(v, \widetilde{\lambda}) - \mathcal{E}_\mu \nu_\ell^{J,t}(v, \widetilde{\lambda})$,

$$\mathcal{P}_\mu(\overline{\mathcal{A}}_\ell^\varepsilon(x)) \leq \mathcal{P}_\mu(\int_0^\infty e^{-v} \overline{\nu}_\ell^{J,t}(v, \widetilde{\lambda}) \mathbb{1}_{\overline{\nu}_\ell^{J,t}(v,\widetilde{\lambda}) > \delta} dv > \varepsilon/2) \tag{4.3.12}$$

$$+ \mathcal{P}_\mu(\int_0^\infty e^{-v} dv > \varepsilon/(2\delta)). \tag{4.3.13}$$

The integral in (4.3.13) is finite and so we can choose $\delta > 0$ small enough so that (4.3.13) is zero. It remains to bound (4.3.12). Using $\mathbb{1}_{\overline{\nu}_\ell^{J,t}(v,\widetilde{\lambda}) > \delta} \leq \overline{\nu}_\ell^{J,t}(v, \widetilde{\lambda})/\delta$ and a first order Chebyshev inequality, it is smaller than

$$\mathcal{P}_\mu(\int_0^\infty e^{-v} (\overline{\nu}_\ell^{J,t}(v, \widetilde{\lambda}))^2 dv > \delta\varepsilon/2) \leq 2/(\delta\varepsilon) \int_0^\infty e^{-v} \mathcal{E}_\mu (\overline{\nu}_\ell^{J,t}(v, \widetilde{\lambda}))^2 dv. \tag{4.3.14}$$

The expectation in (4.3.14) is controlled in the proof of Theorem 1.1 in [40] for $\nu_\ell^{J,t_l,\mu}(v, 1)$, where $1 = (1, \dots, 1)$. Writing for $u, t > 0$,

$$\sigma_\ell^{J,t}(v, \widetilde{\lambda}) \equiv \sum_{j=1}^{k_\ell(t)-1} (\mathcal{P}_\mu(\widetilde{\lambda}_j \overline{Z}_{\ell,j} > u | \mathcal{F}_{\ell,j}))^2, \tag{4.3.15}$$

we get by the same arguments as in the proof of Theorem 1.1 in [40] (cf. (2.9)-(2.13)) that (4.3.14) is smaller than

$$\begin{aligned} &2/(\varepsilon\delta) \int_0^\infty dv e^{-v} \{\mathcal{E}_\mu \sigma_\ell^{J,t_l}(v, \widetilde{\lambda}) + \tfrac{\log a_\ell}{\theta_\ell} (\mathcal{E}_\mu \sigma_\ell^{J,t_l}(v, \widetilde{\lambda}) + \mathcal{E}_\mu \nu_\ell^{J,t_l}(v, \widetilde{\lambda}))\} \\ &\leq 2\lambda_{\max}/(\varepsilon\delta) \int_0^\infty dv e^{-\lambda_{\max} v} \{\mathcal{E}_\mu \sigma_\ell^{J,t_l}(v, 1) + \tfrac{\log a_\ell}{\theta_\ell} (\mathcal{E}_\mu \sigma_\ell^{J,t_l}(v, 1) + \mathcal{E}_\mu \nu_\ell^{J,t_l}(v, 1))\}, \end{aligned} \tag{4.3.16}$$

where we used $k_\ell(t_l) \sup_y \pi_\ell^{t_l,\mu}(y) \leq \frac{\log a_\ell}{\theta_\ell}$ in the first and that $\widetilde{\lambda} \leq \lambda_{\max} \equiv \max_j \widetilde{\lambda}_j < \infty$ in the second step. The integrals in (4.3.16) are Laplace transforms of the measures that map $u > 0$ to $\mathcal{E}_\mu \sigma_\ell^{J,t_l}(u, 1)$ and $\mathcal{E}_\mu \nu_\ell^{J,t_l}(u, 1)$. The Conditions (A-2) and (A-3) of Theorem 1.1 in [40] ask that, $\mathbb{P}$-a.s. for all $u > 0$ and $t > 0$, $\mathcal{E}_\mu \nu_\ell^{J,t}(u, 1) \to t\nu(u, \infty)$ and $\mathcal{E}_\mu \sigma_\ell^{J,t}(u, 1) \to 0$ as $\ell \to \infty$. By Proposition 4.9, where we established that (C-2') $\Rightarrow$ (A-2) and that (C-3') $\Rightarrow$ (A-3), we therefore know that these measures converge $\mathbb{P}$-a.s. for all $u > 0$ and $t > 0$. By continuity of Laplace transforms, we hence know that, $\mathbb{P}$-a.s., for $\varepsilon' \gg \varepsilon\delta$ there exists ℓ large enough such that the left hand side of (4.3.16) is bounded above by

$$2\lambda_{\max}/(\varepsilon\delta)\varepsilon' + \varepsilon'(1 + \int_0^\infty dv e^{-\lambda_{\max} v} t_l \nu(v, \infty)) \leq C\varepsilon', \tag{4.3.17}$$

for some $C \in (0, \infty)$ that is increasing in t_l and in $\lambda_{\max}$. This finishes the proof of (4.3.1).

4.4 Convergence of R

In this Section, we establish assertion (iii) of Theorem 4.1. The proof combines the idea of proof of Theorem 1.6 in [37], which is based on renewal theory for Lévy processes, with the idea of proof of Theorem 1.5 in [40]. First we prove Theorem 4.1 supposing that Conditions (A-1) and (A-2) hold $\mathbb{P}$-a.s. and then comment on the required changes if they hold in $\mathbb{P}$-probability.

We begin with the proof that, $\mathbb{P}$-a.s.,

$$\lim_{\ell\to\infty} \mathcal{P}_\mu(A_\ell(t_w,t)) = R_\infty(t_w,t), \quad \forall t_w > 0,\ t>0 \tag{4.4.1}$$

where,

$$A_\ell(t_w,t) \equiv \{\{\sigma_\ell + S_\ell^b(v), v\geq 0\} \cap (t_w, t_w+t) = \emptyset\}. \tag{4.4.2}$$

By Theorem 1.3 in [37], we know that $\mathbb{P}$-a.s., $\sigma_\ell + S_\ell^b \Rightarrow \sigma + V_\alpha$. Then, since $\mathbb{P}$-a.s. σ is independent of V_α, we know by Theorem 1.6 in [37] that the claim of (4.4.1) holds $\mathbb{P}$-a.s.

Now, let us establish that, $\mathbb{P}$-a.s.,

$$\lim_{\ell\to\infty} |\mathcal{P}_\mu(A_\ell(t_w,t)) - R(c_\ell t_w, c_\ell t)| = 0 \quad \forall t_w>0,\ t>0. \tag{4.4.3}$$

For this, we set $B(c_\ell t_w, c_\ell t) \equiv \{X(c_\ell t_w) = X(c_\ell(t_w+t))\}$. Let us first prove that, $\mathbb{P}$-a.s.,

$$\lim_{\ell\to\infty} \mathcal{P}_\mu(A_\ell(t_w,t), (B(c_\ell t_w, c_\ell t))^c) = 0. \tag{4.4.4}$$

Let $\delta > 0$. Note that, since $S_\ell^b \Rightarrow V_\alpha$ and since $V_\alpha + \sigma$ has $\mathcal{P}$-a.s. diverging paths, there exists $M > 0$ (large, but finite) such that, $\mathbb{P}$-a.s.,

$$\mathcal{P}_\mu(S_\ell^b(M) + \sigma_\ell \leq t_w + t) \leq \delta, \tag{4.4.5}$$

and so with probability at least $1-\delta$, $S_\ell^b + \sigma_\ell$ jumps over (t_w, t_w+t) before time M. By Lemma 4.8 we know that, $\mathbb{P}$-a.s., for c_ℓ large enough,

$$\mathcal{P}_\mu\left(\int_0^{k_\ell(M)} \tau(J(s)) \mathbb{1}_{J(s)\in\mathcal{B}_\ell} > c_\ell \epsilon_\ell^{(1-\alpha)/3}\right) \leq \delta, \tag{4.4.6}$$

and therefore the contribution coming from y such that $\tau(y) \leq c_\ell \epsilon_\ell$ to $Z_{\ell,k}$ is bounded above by $\epsilon_\ell^{(1-\alpha)/3}$. Moreover, writing $T_k = \{x \in \mathcal{T}_\ell : l_{(k+1)\theta_\ell}(x) - l_{k\theta_\ell}(x) > 0\}$, we have by (4.2.38), $\mathbb{P}$-a.s.,

$$\mathcal{P}_\mu\left(\bigcup_{k\leq k_\ell(M)} \{|T_k| \geq 2\} \cup \{\max_{y\sim T_k} \tau(y) > \epsilon_\ell^{-2/\alpha}\}\right) \leq \delta. \tag{4.4.7}$$

By (4.4.6) and (4.4.7), we know that on the event $A_\ell(t_w,t)$ there exists one $x \in \mathbb{Z}^d \cap \mathcal{T}_\ell$ and that there exist v, v' such that $v \leq t_w - \epsilon_\ell^{(1-\alpha)/3}/c_\ell$ and $v' \leq (t_w+t) - \epsilon_\ell^{(1-\alpha)/3}/c_\ell$ such that $X(c_\ell v) = X(c_\ell v') = x$. The event $A_\ell(t_w,t) \cap (B(c_\ell t_w, c_\ell t))^c$ can only be realized when $X(c_\ell s) \neq x$ for $s \in \{t_w, t_w+t\}$. Using the fact $x \in \mathcal{T}_\ell$ and that $\max_{y\sim x} \tau(y) \leq \epsilon_\ell^{-2/\alpha}$, one can prove as in Section 5.1 in [40] (see below eq. (5.9) in Step 1) that with probability larger than $1-\delta$, $X(c_\ell t_w) = X(c_\ell(t_w+t))$. Let us now establish that, $\mathbb{P}$-a.s.,

$$\lim_{\ell\to\infty} \mathcal{P}_\mu((A_\ell(t_w,t))^c, B(c_\ell t_w, c_\ell t)) = 0. \tag{4.4.8}$$

For this, we distinguish whether, on $(A_\ell(t_w,t))^c$, there exists $s > 0$ such that $\sigma_\ell + S_\ell^b(s) \in (t_w, t_w+t)$ or not. Whenever there exists such $s > 0$, one can proceed as in Step 2 in Section 5.1 in [40] to establish the claim of (4.4.8). Let us assume that $\sigma_\ell \in (t_w, t_w+t)$ and $\sigma_\ell + S_\ell^b(s) > t_w + t$ for all $s > 0$. Then,

$$\begin{aligned} &\mathcal{P}_\mu(\sigma_\ell \in (t_w+t-\delta, t_w+t)) \\ \leq\ &\mathcal{P}_\mu(\sigma \in (t_w+t-2\delta/3, t_w+t-\delta/3)) + \mathcal{P}_\mu(|\sigma_\ell - \sigma| > \delta/3), \end{aligned} \tag{4.4.9}$$

which, $\mathbb{P}$-a.s., is bounded above by δ'. Thus, with $\mathcal{P}_\mu$-probability at least $1-\delta'$, $\{\sigma_\ell + S^b_\ell(s) > t_w + t\}$ can only be realized if $Z_{\ell,1} > \delta$ and it suffices to bound

$$\textstyle\sum_{y\in\overline{\mathcal{A}}_\ell} \mu(y)\mathcal{P}_y(Z_{\ell,0} \in (t_w+\delta, t_w+t-\delta), Z_{\ell,1} > \delta). \tag{4.4.10}$$

As in (4.2.24), we bound $\mu(y) \le \mu_1(y) + \mu_2(y)$, for $y \in \overline{\mathcal{A}}_\ell$, and control the contribution coming from μ_i to (4.4.10) separately. We call (I) the contribution from μ_1 and (II) that from μ_2. We first bound (II). From (4.2.18), (4.2.28) and (3.8) of Lemma 3.2 in [40], we know that $\mathbb{P}$-a.s., when $J(0) \in \mathcal{B}_\ell$, then $Z_{\ell,0}$ tends to zero, proving that (II) tends $\mathbb{P}$-a.s. to zero. Let us now control (I). Let $y \in \overline{\mathcal{A}}_\ell \cap \mathcal{T}_\ell$. By (4.2.38), we know that, $\mathbb{P}$-a.s., $Z_{\ell,0} - c_\ell^{-1} l_{\theta_\ell}(y)\tau(y) \le \epsilon_s^{-(1-\alpha)/3}$ and $Z_{\ell,1} - c_\ell^{-1}(l_{2\theta_\ell}(y) - l_{\theta_\ell}(y))\tau(y) \le \epsilon_s^{-(1-\alpha)/3}$. Thus, $\mathbb{P}$-a.s.,

$$(I) \le \textstyle\sum_{y\in\overline{\mathcal{A}}_\ell} \mu_2(y)\mathcal{P}_y\big(l_{\theta_\ell}(y)\tau(y) > c_\ell(t_w + \frac{\delta}{3}), (l_{2\theta_\ell}(y) - l_{\theta_\ell}(y))\tau(y) > c_\ell\frac{\delta}{3}\big). \tag{4.4.11}$$

By (4.2.29) we know that the right hand side of (4.4.11) tends $\mathbb{P}$-a.s. to zero. Hence, the proof of (4.4.8) is finished.

Suppose now that Conditions (A-1) and (A-2) hold in $\mathbb{P}$-probability. The claim of (4.4.1) can be derived from Theorems 1.3 and 1.6 in [37] because both theorems are also formulated for this convergence mode. Also, the proof of (4.4.4) only uses the convergence of S^b_ℓ, and so it does not change with the convergence mode in (A-1) and (A-2). In order to establish (4.4.8), we used the convergence of σ_ℓ to σ to bound $\mathcal{P}_\mu(\sigma_\ell \in (t_w+t-\delta, t_w+t))$ (see (4.4.9)). If Condition (A-1) holds in $\mathbb{P}$-probability we hence get that this tends to zero in $\mathbb{P}$-probability. The remainder of the proof is unchanged.

4.5 A representation

This and the following sections are devoted to the study of correlation functions for the class of initial distributions $\mu_{A,b}$ defined in (4.1.15). As remarked below (4.1.15), when $\mu = \mu_{A,b}$, the law of σ_ℓ converges only in law with respect to the random environment. Since this is not strong enough for the application of Theorem 4.1, we must construct a representation of the random environment in which we can obtain almost sure convergence statements for the initial block. The representation we construct below was first proposed in [37].

Let $(\Omega^E, \mathcal{F}^E, \mathbb{P}^E)$ be a probability space that is independent of $(\Omega, \mathcal{F}, \mathbb{P})$. Let $(E_i, i \in \mathbb{N})$ be a collection of i.i.d. exponential random variables with mean one. For $x \in A$ define the rescaled 'trap' through

$$\gamma_\ell(x) \equiv \ell^{-b/\alpha}\tau(x)^b, \tag{4.5.1}$$

and denote the extreme order statistics of $(\gamma_\ell(x),\ x \in A)$ by $\gamma_{\ell,1} \ge \ldots \ge \gamma_{\ell,\ell}$. A representation of $(\gamma_{\ell,1}, \ldots, \gamma_{\ell,\ell})$ on $(\Omega^E, \mathcal{F}^E, \mathbb{P}^E)$ is then given by $(\boldsymbol{\gamma}_{\ell,1}, \ldots, \boldsymbol{\gamma}_{\ell,\ell})$, where

$$\boldsymbol{\gamma}_{\ell,k} \equiv \ell^{-b/\alpha}\Big(G^{-1}\Big(\textstyle\sum_{i=1}^k E_i\big(\sum_{i=1}^\ell E_i\big)^{-1}\Big)\Big)^b, \quad 1 \le k \le \ell, \tag{4.5.2}$$

where $G(u) = \mathbb{P}(\tau(0) > u)$ is the tail distribution function of $\tau(0)$ and G^{-1} its right-continuous inverse. Indeed, by Lemma 6.1 in [37],

$$(\boldsymbol{\gamma}_{\ell,1}, \ldots, \boldsymbol{\gamma}_{\ell,\ell}) \stackrel{d}{=} (\gamma_{\ell,1}, \ldots, \gamma_{\ell,\ell}). \tag{4.5.3}$$

In order to construct a representation of σ_ℓ in which we can make almost sure convergence statements, we also need to represent the random variables $\tau(x)$ for which $x \notin A$ and construct a representation for J. Let $(\mathbf{\Omega}, \boldsymbol{\mathcal{F}}, \mathbf{P})$ be the product space of $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\Omega^E, \mathcal{F}^E, \mathbb{P}^E)$. Assume that the elements of A are ordered alphabetically, say $A = \{x_1, \dots, x_\ell\}$ and let $\pi : \{1, \dots, \ell\} \to \{1, \dots, \ell\}$ be the random permutation such that $\gamma_\ell(x_i) = \gamma_{\ell,\pi(i)}$. (Since the random environment is a collection of i.i.d. random variables, π is distributed according to the uniform distribution on all permutations on $\{1, \dots, \ell\}$.) Let us now define our key objects on $(\mathbf{\Omega}, \boldsymbol{\mathcal{F}}, \mathbf{P})$. We use bold fonts to denote objects expressed in this representation, i.e. the random environment is denoted by $\boldsymbol{\tau}$ and given by

$$\boldsymbol{\tau}(x) = \ell^{1/\alpha}\gamma_{\ell,\pi(k)}^{1/b}\mathbb{1}_{x=x_k\in A} + \tau(x)\mathbb{1}_{x\notin A}, \quad x \in \mathbb{Z}^d. \tag{4.5.4}$$

Notice that since π has a uniform distribution on permutations on $\{1, \dots, \ell\}$, $(\boldsymbol{\tau}(x), x \in \mathbb{Z}^d)$ is a collection of i.i.d. random variables with distribution function $\mathbf{P}(\boldsymbol{\tau}(x) > u) = G(u) = \mathbb{P}(\tau(0) > u)$. The dynamics that we study is the same as before, substituting the new random environment, $\boldsymbol{\tau}$, for the original one, τ. In particular, $\mathbf{J}$ is the continuous time Markov chain with infinitesimal generator given by

$$\boldsymbol{\lambda}(y, z) \equiv \left[\boldsymbol{\tau}(y)\boldsymbol{\tau}(z)\right]^\theta, \tag{4.5.5}$$

if $y \sim z$ and zero else. The initial distribution is given by

$$\boldsymbol{\mu}_{A,b}(x_k) = \boldsymbol{\mu}_{A,b}(\pi(k)) \equiv \boldsymbol{\gamma}_{\ell,\pi(k)}\left(\textstyle\sum_{i=1}^{\ell}\boldsymbol{\gamma}_{\ell,i}\right)^{-1}. \tag{4.5.6}$$

For a time scale c_ℓ, an auxiliary time scale a_ℓ, and a sequence $\theta_\ell \ll a_\ell$ we define the block variables through

$$\mathbf{Z}_{\ell,i} \equiv c_\ell^{-1}\sum_{y\in\mathbb{Z}^d}\boldsymbol{\tau}(y)\left[l_{(i+1)\theta_\ell}(y) - l_{i\theta_\ell}(y)\right], \quad i \geq 0, \tag{4.5.7}$$

where $\mathbf{l}_t(x) \equiv \int_0^t \mathbb{1}_{\mathbf{J}(v)=x}dv$ denotes the local time of $\mathbf{J}$ in x. By analogy to (4.1.10) and (4.1.11) the initial block is given by

$$\boldsymbol{\sigma}_\ell \equiv \mathbf{Z}_{\ell,0}, \tag{4.5.8}$$

and the pure clock is defined through

$$\mathbf{S}_\ell^b(t) \equiv \sum_{i=1}^{k_\ell(t)-1}\mathbf{Z}_{\ell,i}, \quad t > 0. \tag{4.5.9}$$

In the following, we study $\boldsymbol{\sigma}_\ell$ and $\mathbf{S}_\ell^b$ instead of σ_ℓ and S_ℓ^b and verify that the conditions of Theorem 4.1 are satisfied $\mathbf{P}$-a.s. In order to derive from this convergence statements in $(\Omega, \mathcal{F}, \mathbb{P})$, we use the following lemma, whose proof is taken form [38].

Let $\{X_{n,k}^1, 1 \leq k \leq n\}$ be an array of random variables defined on $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ and let $\{X_{n,k}^2, k \in \mathbb{N} \setminus \{1, \dots, n\}\}$ be an array of random variables defined on $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$, independent of $\{X_{n,k}^1, 1 \leq k \leq n\}$. Let $\{\hat{X}_{n,k}^1, 1 \leq k \leq n\}$ be an array defined on $(\hat{\Omega}_1, \hat{\mathcal{F}}_1, \hat{\mathbb{P}}_1)$ satisfying

$$(X_{n,1}^1, \dots, X_{n,n}^1) \stackrel{d}{=} (\hat{X}_{n,1}^1, \dots, \hat{X}_{n,n}^1). \tag{4.5.10}$$

Lemma 4.15. *Let $g_n(\{x^1\}, \{x^2\})$ be a sequence of positive, bounded, real valued functions on $\mathbb{R}^n \times \mathbb{R}^{\mathbb{N}}$. If $\lim_{n\to\infty} g_n(\{\hat{X}_{n,k}^1\}, \{X_{n,k}^2\}) = \hat{Y}$, $\hat{\mathbb{P}}_1$-a.s. $\times$ $\mathbb{P}_2$-a.s., where $\hat{Y}$ is defined on $(\hat{\Omega}_1, \hat{\mathcal{F}}_1, \hat{\mathbb{P}}_1) \times (\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$, then, $\lim_{n\to\infty} g_n(\{X_{n,k}^1\}, \{X_{n,k}^2\}) = Y$, in $\mathbb{P}_1$-law $\times$ $\mathbb{P}_2$-law, where Y is defined on $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1) \times (\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$, and $Y \stackrel{d}{=} \hat{Y}$.*

Proof. The assumptions in Lemma 4.15 state that there exist $\hat{\Omega}_1' \subseteq \hat{\Omega}_1$ and $\Omega_2' \subseteq \Omega_2$ with $\hat{\mathbb{P}}_1(\hat{\Omega}_1') = \mathbb{P}_2(\Omega_2') = 1$ such that, for all $(\omega_1, \omega_2) \in \hat{\Omega}_1' \times \Omega_2'$

$$\lim_{n\to\infty} g_n(\{\hat{X}^1_{n,k}(\omega_1)\}, \{X^2_{n,k}(\omega_2)\}) = \hat{Y}. \tag{4.5.11}$$

Let f be a bounded and continuous function. Set $Z_n(\omega) \equiv f(g_n(\{\hat{X}^1_{n,k}(\omega)\}, \{X^2_{n,k}\}), \omega \in \hat{\Omega}_1$. Since f is bounded and continuous, we get by the dominated convergence theorem that $\lim_{n\to\infty} \mathbb{E}_2 Z_n(\omega_1) = \mathbb{E}_2 f(\hat{Y}(\omega_1))$ for all $\omega_1 \in \hat{\Omega}_1'$ and for $\bar{Z}_n(\omega) \equiv \mathbb{E}_2 Z_n(\omega)$, $\omega \in \hat{\Omega}_1$, that $\lim_{n\to\infty} \hat{\mathbb{E}}_1 \bar{Z}_n = \hat{\mathbb{E}}_1\mathbb{E}_2 f(\hat{Y})$. Thus, for all bounded and continuous functions f,

$$\lim_{n\to\infty} \hat{\mathbb{E}}_1\mathbb{E}_2 f(g_n(\{\hat{X}^1_{n,k}\}, \{X^2_{n,k}\})) = \hat{\mathbb{E}}_1\mathbb{E}_2 f(\hat{Y}). \tag{4.5.12}$$

By (4.5.10), for all $n \geq 1$, $\hat{\mathbb{E}}_1\mathbb{E}_2 f(g_n(\{\hat{X}^1_{n,k}\}, \{X^2_{n,k}\})) = \mathbb{E}_1\mathbb{E}_2 f(g_n(\{X^1_{n,k}\}, \{X^2_{n,k}\}))$. Moreover, $\hat{\mathbb{E}}_1\mathbb{E}_2 f(\hat{Y}) = \mathbb{E}_1\mathbb{E}_2 f(Y)$. We conclude by Portmanteau's Lemma that $g_n(\{X^1_{n,k}\}, \{X^2_{n,k}\})$ converges in $\mathbb{P}_1 \times \mathbb{P}_2$-law to Y as $n \to \infty$. $\square$

4.6 The initial block

In this section we study $\boldsymbol{\sigma}_\ell$ as defined in (4.5.8). Let us begin with a key property of the measure $\boldsymbol{\mu}_{A,b}$, which rejoins the earlier stated fact that, since $\alpha < b$, $(\boldsymbol{\mu}_{A,b}(k))_{k=1}^{\ell}$ converges to a Poisson Dirichlet distribution. This property is that, in expectation with respect to the random environment, almost all mass of the measure $\boldsymbol{\mu}_{A,b}$ is contained in a set $\boldsymbol{\mathcal{T}}_\ell \subseteq \{1, \dots \ell\}$ consisting of the 'top' traps. In order to construct this set, let

$$\bar{\epsilon}_\ell \equiv \bar{\epsilon}_\ell(d) = (\log c_\ell)^{-1} \mathbb{1}_{d\leq 2} + (\log c_\ell)^{-3} \mathbb{1}_{d\geq 3}, \tag{4.6.1}$$

Then, the set of 'top' traps is defined through

$$\boldsymbol{\mathcal{T}}_\ell \equiv \{k : \boldsymbol{\gamma}_{\ell,k}^{1/b} > \bar{\epsilon}_\ell\}. \tag{4.6.2}$$

The following lemma, shows that the expectation of $\boldsymbol{\mu}_{A,b}(\mathbf{A} \setminus \boldsymbol{\mathcal{T}}_\ell)$ tends to zero as $\ell \to \infty$.

Lemma 4.16. *There exists $c \in (0, \infty)$ such that we have for all $\delta > 0$*

$$\mathbf{E}\boldsymbol{\mu}_{A,b}(\mathbf{A} \setminus \boldsymbol{\mathcal{T}}_\ell) = \mathbb{E}\mu_{A,b}(A \setminus \mathcal{T}_\ell) \leq c\bar{\epsilon}_\ell^{(b-\alpha)/4}. \tag{4.6.3}$$

Proof. We bound $\mu_{A,b}(A \setminus \mathcal{T}_\ell) \leq \mu_{A,b}(A \setminus \mathcal{T}_\ell)\mathbb{1}_{B_\ell} + \mathbb{1}_{(B_\ell)^c}$, where

$$B_\ell \equiv \{\textstyle\sum_{y\in A} \mathbb{1}_{\tau(y)>\ell^{1/\alpha}\bar{\epsilon}_\ell^{3/4}} > \frac{1}{2}|A|\mathbb{P}(\tau(y) > \ell^{1/\alpha}\bar{\epsilon}_\ell^{3/4}).\}. \tag{4.6.4}$$

Let us first establish that $\mathbb{P}(B_\ell^c) \leq \bar{\epsilon}_\ell^{1/4(b-\alpha)}$. By Bennett's inequality,

$$\mathbb{P}(B_\ell^c) \leq \mathbb{P}\left(\left|\textstyle\sum_{y\in A} \mathbb{1}_{\tau(y)>\ell^{1/\alpha}\bar{\epsilon}_\ell^{3/4}} - \bar{\epsilon}_\ell^{-3\alpha/4})\right| > \frac{1}{2}\bar{\epsilon}_\ell^{-3\alpha/4}\right) \leq \exp(-\tfrac{1}{16}\bar{\epsilon}_\ell^{-3\alpha/4}), \tag{4.6.5}$$

which is smaller than $\bar{\epsilon}_\ell^{1/4(b-\alpha)}$. It remains to control $\mu_{A,b}(A \setminus \mathcal{T}_\ell)\mathbb{1}_{B_\ell}$. On B_ℓ we have

$$\begin{aligned}\mu_{A,b}(A \setminus \mathcal{T}_\ell) &= \sum_{x\in A} \frac{\tau(x)^b \mathbb{1}_{\tau(x)\leq \ell^{1/\alpha}\bar{\epsilon}_\ell}}{\sum_{y\in A}\tau(y)^b} \leq \sum_{x\in A} \frac{\tau(x)^b \mathbb{1}_{\tau(x)\leq \ell^{1/\alpha}\bar{\epsilon}_\ell}}{\sum_{y\in A}\tau(y)^b \mathbb{1}_{\tau(y)>\ell^{1/\alpha}\bar{\epsilon}_\ell^{3/4}}} \\ &\leq \tfrac{1}{2}\ell^{(\alpha-b)/\alpha}\bar{\epsilon}_\ell^{3/4(\alpha-b)}|A|^{-1}\textstyle\sum_{x\in A}\tau(x)^b\mathbb{1}_{\tau(x)\leq \ell^{1/\alpha}\bar{\epsilon}_\ell}.\end{aligned} \tag{4.6.6}$$

Each summand on the right hand side of (4.6.6) satisfies $\mathbb{E}\tau(x)^b\mathbb{1}_{\tau(x)\leq \ell^{1/\alpha}\bar{\epsilon}_\ell} \leq c'(\ell^{1/\alpha}\bar{\epsilon}_\ell)^{b-\alpha}$, where $c' \in (0, \infty)$. Therefore,

$$\mathbb{E}(\mu_{A,b}(A \setminus \mathcal{T}_\ell)\mathbb{1}_{B_\ell}) \leq \tfrac{1}{2}c'\,\ell^{(\alpha-b+b-\alpha)/\alpha}\,\bar{\epsilon}_\ell^{-3b/4+3\alpha/4+b-\alpha} = \tfrac{1}{2}c'\,\bar{\epsilon}_\ell^{1/4(b-\alpha)}, \tag{4.6.7}$$

as desired. The proof of Lemma 4.16 is finished. $\square$

4.6.1 Convergence of the initial block on intermediate time scales

In this section we study the initial block $\boldsymbol{\sigma}_\ell^{int}$ when we observe the process on the intermediate time scale, that is when c_ℓ^{int} is given through (4.1.18). Specifically, this section is devoted to the proof of the following proposition.

Proposition 4.17. *Let c_ℓ be given by* (4.1.18). *There exists a collection of i.i.d. random variables $\mathbf{Y}$ such that $\mathbf{E}(\mathbf{Y}_1)^\kappa < \infty$ for all $\kappa \in \mathbb{R}$, and a Poisson random measure $\boldsymbol{\Gamma}$ with intensity measure $\nu_{\alpha/b}$, that are independent of each other and are such that the following holds. Let σ be a random variable on $(0,\infty)$ with distribution function $\mathbf{P}(\boldsymbol{\sigma} \le u) = F_{\mathbf{Y},0}(ut_w^{-\delta})$, $u > 0$, where $F_{\mathbf{Y},0}$ is given by* (4.1.17). *Then, $\mathbf{P}$-a.s., $\boldsymbol{\sigma}_\ell \Rightarrow \boldsymbol{\sigma}$.*

The proof of Proposition 4.17 comes in two steps. First we show that we can, up to a $\mathbf{P}$-a.s. small enough error, approximate $\boldsymbol{\sigma}_\ell$ by another random variable, $\overline{\boldsymbol{\sigma}}_\ell$, which consists only of $\gamma_{\ell,\pi(k)}$ for which $\pi(k) \in \boldsymbol{\mathcal{T}}_\ell$. In the second step we prove that, $\mathbf{P}$-a.s., $\overline{\boldsymbol{\sigma}}_\ell \Rightarrow \boldsymbol{\sigma}$. The approximation for $\boldsymbol{\sigma}_\ell$ is given by

$$\overline{\boldsymbol{\sigma}}_\ell \equiv (c_\ell^{int})^{-1}\ell^{1/\alpha} \textstyle\sum_{\pi(k)\in\boldsymbol{\mathcal{T}}_\ell} \boldsymbol{\gamma}_{\ell,\pi(k)}^{1/b}\ \mathbb{1}_{J(0)=x_k}\mathbf{l}_{\theta_\ell^{int}}(x_k)\mathbb{1}_{\mathcal{D}_\ell^c}, \tag{4.6.8}$$

where $\mathcal{D}_\ell \equiv \mathcal{D}_\ell^1 \cup \mathcal{D}_\ell^2$ is the union of the following two sets

$$\begin{aligned} \mathcal{D}_\ell^1 &\equiv \{\boldsymbol{\omega}\in\boldsymbol{\Omega} : \exists\pi(k)\in\boldsymbol{\mathcal{T}}_\ell :\ \exists y\in\mathbb{Z}^d : 0<|y-x_k|\le 2 : \tau(y) > \theta_\ell^{int}\}, &(4.6.9)\\ \mathcal{D}_\ell^2 &\equiv \{\boldsymbol{\omega}\in\boldsymbol{\Omega} : \exists\pi(k),\pi(l)\in\boldsymbol{\mathcal{T}}_\ell : B_{\overline{\theta}_\ell^{int}}(x_k)\cap B_{\overline{\theta}_\ell^{int}}(x_l)\neq\emptyset\}. &(4.6.10)\end{aligned}$$

Notice that $\mathcal{D}_\ell$ is a "bad" event because on $\mathcal{D}_\ell$ the top traps are not separated from each other. However, we prove in the lemma below that, $\mathbf{P}$-a.s., $\mathbb{1}_{\mathcal{D}_\ell^1}$ and $\mathbb{1}_{\mathcal{D}_\ell^2}$ tend to zero. To shorten notation, we write for the remainder of this section $c_\ell \equiv c_\ell^{int}$, and similarly for the other sequences.

The following lemma states that, $\mathbf{P}$-a.s., $|\boldsymbol{\sigma}_\ell - \overline{\boldsymbol{\sigma}}_\ell|$ vanishes in $\mathcal{P}_{\boldsymbol{\mu}_{A,b}}$-probability.

Lemma 4.18. *For all $\delta > 0$, $\mathbf{P}$-a.s., $\lim_{\varepsilon\to 0}\limsup_{\ell\to\infty}\mathcal{P}_{\boldsymbol{\mu}_{A,b}}(|\boldsymbol{\sigma}_\ell - \overline{\boldsymbol{\sigma}}_\ell| > \varepsilon) = 0$.*

Proof. Fix $\delta > 0$ and $\varepsilon > 0$. Then,

$$\begin{aligned} &\mathcal{P}_{\boldsymbol{\mu}_{A,b}}(|\boldsymbol{\sigma}_\ell - \overline{\boldsymbol{\sigma}}_\ell| > \varepsilon)\\ \le\ & \boldsymbol{\mu}_{A,b}(\boldsymbol{\mathcal{T}}_\ell\setminus\mathbf{A}) + \textstyle\sum_{\pi(k)\in\boldsymbol{\mathcal{T}}_\ell}\boldsymbol{\mu}_{A,b}(\pi(k)|\boldsymbol{\mathcal{T}}_\ell)\mathcal{P}_{x_k}(|\boldsymbol{\sigma}_\ell - \overline{\boldsymbol{\sigma}}_\ell| > \varepsilon). \end{aligned} \tag{4.6.11}$$

By Lemma 4.16 and Chebyshev's inequality, the first term of (4.6.11) tends, $\mathbf{P}$-a.s., to zero. We bound the second term from above by

$$\mathbb{1}_{\mathcal{D}_\ell} + \mathbb{1}_{\mathcal{D}_\ell^c}\boldsymbol{\mu}_{A,b}(\pi(k)|\boldsymbol{\mathcal{T}}_\ell)\mathcal{P}_{x_k}(|\boldsymbol{\sigma}_\ell - \overline{\boldsymbol{\sigma}}_\ell| > \varepsilon). \tag{4.6.12}$$

The first term in (4.6.12) is in expectation smaller than

$$\begin{aligned} \mathbf{P}(\mathcal{D}_\ell) \le \mathbf{P}(\mathcal{D}_\ell^1) + \mathbf{P}(\mathcal{D}_\ell^2) &\le \ell\mathbf{P}(0\in\boldsymbol{\mathcal{T}}_\ell)(4d\mathbb{P}(\tau(0) > \theta_\ell) + |B_{\overline{\theta}_\ell}(0)|\mathbf{P}(0\in\boldsymbol{\mathcal{T}}_\ell)\\ &\le \overline{\epsilon}_\ell^{-\alpha}\theta_\ell^{-\alpha} + \overline{\epsilon}_\ell^{-2\alpha}\ell^{-2}|\overline{\mathcal{A}}_\ell|\overline{\theta}_\ell^d \end{aligned} \tag{4.6.13}$$

which by (4.1.14) and (4.6.1) tends to zero. Let us now to control the second term in (4.6.12). On the event $\mathcal{D}_\ell^c$ the second summand in (4.6.12) is bounded by

$$\begin{aligned} &\textstyle\sum_{\pi(k)\in\boldsymbol{\mathcal{T}}_\ell}\boldsymbol{\mu}_{A,b}(\pi(k)|\boldsymbol{\mathcal{T}}_\ell)\mathcal{P}_{x_k}\big(\int_0^{\theta_\ell}\tau(J(s))\mathbb{1}_{\tau(J(s))\le\ell^{1/\alpha}\overline{\epsilon}_\ell}ds > c_\ell\varepsilon\big) &(4.6.14)\\ +\ &\textstyle\sum_{\pi(k)\in\boldsymbol{\mathcal{T}}_\ell}\mathcal{P}_{x_k}(\exists y\neq x :\ \exists s\le\theta_\ell :\ J(s) = y,\ \tau(y) > \ell^{1/\alpha}\overline{\epsilon}_\ell) \equiv (I) + (II). &(4.6.15)\end{aligned}$$

We bound (I) and (II) separately. Let us begin with (I). Note that by construction, $c_\ell \epsilon_\ell = \ell^{1/\alpha}\overline{\epsilon}_\ell$, and so $\{\tau(J(s)) \leq \ell^{1/\alpha}\overline{\epsilon}_\ell\} = \{J(s) \in \mathcal{T}_\ell\}$, where $\mathcal{T}_\ell$ is defined below (4.6.1). Therefore, one can show as in (4.2.18), that (I) vanishes $\mathbf{P}$-a.s. It remains to bound (II). We have that

$$(II) \leq \textstyle\sum_{\pi(k)\in\boldsymbol{\mathcal{T}}_\ell}\big(\sum_{0<|y-x_k|\leq\overline{\theta}_\ell} \mathbb{1}_{\tau(y)>c_\ell\epsilon_\ell} + \mathcal{P}_{x_k}(\eta(B_{\overline{\theta}_\ell}(x_k) \leq \theta_\ell)\big). \tag{4.6.16}$$

By (3.8) of Lemma 3.2 in [40], the second summand in (4.6.16) is smaller than $e^{-c_4(\log c_\ell)^2}$ and hence tends to zero. Since $\tau(y)$ is independent of $\{\pi(k) \in \boldsymbol{\mathcal{T}}_\ell\}$, the first term of (4.6.16) is in expectation bounded above by $\ell(c_\ell\epsilon_\ell)^{-2\alpha}\overline{\theta}_\ell^d$. Thus, by a first order Chebyshev inequality we may conclude that (4.6.16) vanishes $\mathbf{P}$-a.s. and the proof of Lemma 4.18 is finished. □

Let us now establish that, $\mathbf{P}$-a.s., $\overline{\boldsymbol{\sigma}}_\ell \Rightarrow \boldsymbol{\sigma}$ where $\boldsymbol{\sigma}$ has distribution function $F_{\mathbf{Y},0}$ as in Proposition 4.17. We define $\boldsymbol{\Gamma}$ on $(\boldsymbol{\Omega}, \boldsymbol{\mathcal{F}}, \mathbf{P})$ through

$$\boldsymbol{\Gamma} \equiv \textstyle\sum_{k\in\mathbb{N}} \delta_{\gamma_k}, \quad \text{where} \quad \gamma_k \equiv \Big(\sum_{i=1}^k E_i\Big)^{-b/\alpha}. \tag{4.6.17}$$

For the definition of $\mathbf{Y}$ on $(\boldsymbol{\Omega}, \boldsymbol{\mathcal{F}}, \mathbf{P})$, we distinguish whether $d = 2$ and $d \geq 3$. By assumption, when $d = 2$, J is independent of the random environment and the elements of $\mathbf{Y}$ are given by

$$\mathbf{Y}_k = \lim_{\ell\to\infty} (\ell^{1/\alpha}/(t_w^\delta c_\ell)) E_0 \widehat{l}_{\eta(B_{\sqrt{\theta_\ell}}(0))}(0) = \lim_{\ell\to\infty} (\log c_\ell)^{-1} E_0 \widehat{l}_{\eta(B_{\sqrt{\theta_\ell}}(0))}(0). \tag{4.6.18}$$

It is well-known that this limit exists and is strictly positive. When $d \geq 3$, let $\mathbf{Y}$ be a collection of independent copies of

$$\mathbf{Y}_1 = \lim\nolimits_{\ell\to\infty}(\ell^{1/\alpha}/(t_w^\delta c_\ell))\widetilde{g}_{B_{\sqrt{\theta_\ell}}(0))^{-1}}, \tag{4.6.19}$$

where

$$(\widetilde{g}_{B_{\sqrt{\theta_\ell}}(0)})^{-1} \equiv \inf\left\{\tfrac{1}{2}\textstyle\sum_{x\sim z}\lambda(x,z)(f(x)-f(z))^2 : f|_0 = 1, f|_{B_{\sqrt{\theta_\ell}}(0)} = 0\right\}. \tag{4.6.20}$$

Since $\widetilde{g}_{B_{\sqrt{\theta_\ell}}}(0)$ depends only on $y \in B_{\sqrt{\theta_\ell}}(0)$, one can show that $\mathbf{P}$-a.s., $\widetilde{g}_{B_{\sqrt{\theta_\ell}}}(0)$ is asymptotically independent of $\boldsymbol{\Gamma}$. By Lemma 3.4 in [2] we know that $\mathbf{Y}_1 \leq c < \infty$ and that

$$\mathbf{P}(\mathbf{Y}_1 \leq v) \leq c_1 \exp(-c_2 v^{-(d-2)/3}). \tag{4.6.21}$$

Notice that therefore $\mathbf{E}\mathbf{Y}_1^\kappa < \infty$ for all $\kappa \in \mathbb{R}$.

The following lemma is a first step in proving that the distribution function of $\overline{\boldsymbol{\sigma}}_\ell$ converges to $F_{\mathbf{Y},0}$.

Lemma 4.19. *For all $u > 0$, $\delta > 0$, and $\varepsilon > 0$ there exists ℓ_0 such that, $\mathbf{P}$-a.s., for $\ell \geq \ell_0$*

$$\big|\mathcal{P}_{\boldsymbol{\mu}_{A,b}}(\overline{\boldsymbol{\sigma}}_\ell > u) - \big(1 - \mathbf{F}_\ell(u/t_w^\delta)\big)\big| < \varepsilon, \tag{4.6.22}$$

where for $\rho \geq 0$ we set

$$\mathbf{F}_\ell(u/t_w^\delta) \equiv 1 - \textstyle\sum_{k=1}^\ell \boldsymbol{\mu}_{A,b}(k|\boldsymbol{\mathcal{T}}_\ell)\exp\big(-u/\big(t_w^\delta \gamma_{\ell,k}^{1/b}\mathbf{Y}_k^1\big)\big)\big| < \varepsilon. \tag{4.6.23}$$

Proof. Let $u > 0$, $\delta > 0$ and $\varepsilon > 0$. By a first order Chebyshev inequality we have for $\boldsymbol{\mathcal{T}}_\ell^J \equiv \{J(0) = x_k : \pi(k) \in \boldsymbol{\mathcal{T}}_\ell\}$

$$\mathbf{P}\big(|\mathcal{P}_{\boldsymbol{\mu}_{A,b}}(\overline{\boldsymbol{\sigma}}_\ell > u) - \mathcal{P}_{\boldsymbol{\mu}_{A,b}}(\overline{\boldsymbol{\sigma}}_\ell > u|\boldsymbol{\mathcal{T}}_\ell^J)| > \varepsilon\big) \leq \tfrac{1}{\varepsilon}\mathbf{E}\big[\boldsymbol{\mu}_{A,b}(\mathbf{A}\setminus\boldsymbol{\mathcal{T}}_\ell)/\boldsymbol{\mu}_{A,b}(\mathbf{A}_)\big]. \tag{4.6.24}$$

As in Lemma 4.16 one can show that this tends to zero faster than $\bar{\epsilon}_\ell^{(b-\alpha)/4}$. Hence, Borel-Cantelli Lemma implies that, $\mathbf{P}$-a.s., for ℓ large enough we have

$$|\mathcal{P}_{\boldsymbol{\mu}_{A,b}}(\overline{\boldsymbol{\sigma}}_\ell > u) - \mathcal{P}_{\boldsymbol{\mu}_{A,b}}(\overline{\boldsymbol{\sigma}}_\ell > u|\boldsymbol{\mathcal{T}}_\ell^J)| < \varepsilon. \tag{4.6.25}$$

Notice that,

$$\mathcal{P}_{\boldsymbol{\mu}_{A,b}}(\overline{\boldsymbol{\sigma}}_\ell > u|\boldsymbol{\mathcal{T}}_\ell^J) = \textstyle\sum_k \boldsymbol{\mu}_{A,b}(\pi(k)|\boldsymbol{\mathcal{T}}_\ell)\mathcal{P}_{x_k}\big(\boldsymbol{\gamma}_{\ell,\pi(k)}^{1/b}\mathbf{l}_{\theta_\ell}(x_k) > u\frac{c_\ell}{\ell^{1/\alpha}}\big)\mathbb{1}_{\mathcal{D}_\ell^c}, \tag{4.6.26}$$

i.e. $\mathcal{P}_{\boldsymbol{\mu}_{A,b}}(\overline{\boldsymbol{\sigma}}_\ell > u|\mathbf{A}_\ell^1)$ is equal to zero on $\mathcal{D}_\ell$. Bounding $1 - \mathbf{F}_\ell(u/t_w^\delta)$ by one on $\mathcal{D}_\ell$ and using (4.6.13), we see that (4.6.22) holds on $\mathcal{D}_\ell$ for ℓ large enough. From now on, let $\boldsymbol{\omega} \in \mathcal{D}_\ell^c$.

Let us construct $\mathbf{P}$-a.s. upper and lower bounds for (4.6.26) and show that both tend to the sum in (4.6.22) as $\ell \to \infty$. We begin by constructing bounds for the local time $\mathbf{l}_{\theta_\ell}(x_k)$ for $\pi(k) \in \boldsymbol{\mathcal{T}}_\ell$. For the lower bound, define $B_{\ell,k}^1 \equiv B_{\sqrt{\theta_\ell}(\log\theta_\ell)^{-2}}(x_k)$, $\pi(k) \in \boldsymbol{\mathcal{T}}_\ell$. By (3.9) of Lemma 3.2 in [40] we know that $B_{\ell,k}^1$ is exited before θ_ℓ with probability larger than $1 - \exp(-c_4(\log\theta_\ell)^2)$ and so

$$\mathbf{P}\big(\textstyle\sum_{\pi(k)\in\boldsymbol{\mathcal{T}}_\ell} P_{x_k}(\eta(B_{\ell,k}^1) \le \theta_\ell) > \varepsilon\big) \le \mathbf{E}|\boldsymbol{\mathcal{T}}_\ell|e^{-c_4(\log\theta_\ell)^2}, \tag{4.6.27}$$

which, since $\mathbf{E}|\boldsymbol{\mathcal{T}}_\ell| = \ell c_\ell^{-\alpha}\epsilon_\ell^{-\alpha} \le \epsilon_\ell^{-\alpha/2}$, tends to zero. Hence, $\mathbf{P}$-a.s., for ℓ large enough,

$$\begin{aligned}\mathcal{P}_{\boldsymbol{\mu}_{A,b}}(\overline{\boldsymbol{\sigma}}_\ell > u|\boldsymbol{\mathcal{T}}_\ell^J) &\ge \textstyle\sum_k \boldsymbol{\mu}_{A,b}(\pi(k)|\boldsymbol{\mathcal{T}}_\ell)\mathcal{P}_{x_k}\big(\boldsymbol{\gamma}_{\ell,\pi(k)}^{1/b}\mathbf{l}_{\eta(B_{\ell,k}^1)}(x_k) > uc_\ell\ell^{-1/\alpha}\big) - \varepsilon \\ &= \textstyle\sum_k \boldsymbol{\mu}_{A,b}(\pi(k)|\boldsymbol{\mathcal{T}}_\ell)\exp\{-uc_\ell/[\ell^{1/\alpha}\boldsymbol{\gamma}_{\ell,\pi(k)}^{1/b}E_{x_k}\mathbf{l}_{\eta(B_{\ell,k}^1)}(x_k)]\} - \varepsilon,\end{aligned} \tag{4.6.28}$$

where we used in the second step that $\mathbf{l}_{\eta(B_{\ell,k}^1)}(x_k)$ is exponentially distributed. For the upper bound, we substitute, by (3.8) of Lemma 3.2 in [40], $\mathbf{l}_{\eta(B_{\ell,k}^2)}(x_k)$ for $\mathbf{l}_{\theta_\ell}(x_k)$, where $B_{\ell,k}^2 \equiv B_{\sqrt{\theta_\ell}\log\theta_\ell}(x_k)$. Thus, $\mathbf{P}$-a.s., for ℓ large enough

$$\mathcal{P}_{\boldsymbol{\mu}_{A,b}}(\overline{\boldsymbol{\sigma}}_\ell > u|\boldsymbol{\mathcal{T}}_\ell^J) \le \textstyle\sum_k \boldsymbol{\mu}_{A,b}(\pi(k)|\boldsymbol{\mathcal{T}}_\ell)\exp\{-c_\ell u/[\ell^{1/\alpha}\boldsymbol{\gamma}_{\ell,\pi(k)}^{1/b}E_{x_k}\mathbf{l}_{\eta(B_{\ell,k}^2)}(x_k)]\} + \varepsilon. \tag{4.6.29}$$

The proof of Lemma 4.19 will be finished, once we have established that, $\mathbf{P}$-a.s., we may replace $E_{x_k}\mathbf{l}_{\eta(B_{\ell,k}^i)}(x_k)$ for $i = 1, 2$ and ℓ large enough by $\mathbf{Y}_k$. For this, let us distinguish whether $d = 2$ or $d \ge 3$. Let $d = 2$ first. Then, J is not random in the random environment. In particular, $\{E_{x_k}\mathbf{l}_{\eta(B_{\ell,k}^i)}(x_k), \pi(k) \in \boldsymbol{\mathcal{T}}_\ell\}$ is for $i = 1, 2$ independent of $\{\gamma_{\ell,k}, k \in \boldsymbol{\mathcal{T}}_\ell\}$. Moreover, one can check that

$$E_{x_k}\mathbf{l}_{\eta(B_{\ell,k}^i)}(x_k) = E_0\mathbf{l}_{\eta(B_{\ell,0}^i)}(0), \quad i = 1, 2. \tag{4.6.30}$$

As remarked in (4.6.18),

$$(1-\varepsilon)\mathbf{Y}_k \le t_w^\delta c_\ell^{-1}\ell^{1/\alpha}E_0\big(\mathbf{l}_{\eta(B_\ell^1)}(0)\big) \le t_w^\delta c_\ell^{-1}\ell^{1/\alpha}E_0\big(\mathbf{l}_{\eta(B_\ell^2)}(0)\big) \le (1+\varepsilon)\mathbf{Y}_k. \tag{4.6.31}$$

Using the continuity of the exponential function in (4.6.28) and (4.6.29) we get, $\mathbf{P}$-a.s., for ℓ large enough

$$|\mathcal{P}_{\boldsymbol{\mu}_{A,b}}(\overline{\boldsymbol{\sigma}}_\ell > u) - \textstyle\sum_k \boldsymbol{\mu}_{A,b}(k|\boldsymbol{\mathcal{T}}_\ell)\exp\big(-u/(t_w^\delta\boldsymbol{\gamma}_{\ell,k}^{1/b}\mathbf{Y}_k)\big)| < \varepsilon, \tag{4.6.32}$$

as claimed in (4.6.22). The proof of Lemma 4.19 is finished for $d = 2$.

Let $d \ge 3$. Then J is random in the random environment and it is a priori not clear that the $\widetilde{\mathbf{Y}}_k$'s should be i.i.d. and independent of $\{\gamma_{\ell,k}, k \in \boldsymbol{\mathcal{T}}_\ell\}$. However, we know from Lemma 3.5 in [2] that, $\mathbf{P}$-a.s., uniformly in $\pi(k) \in \boldsymbol{\mathcal{T}}_\ell$,

$$(1-\varepsilon)\widetilde{\mathbf{Y}}_k \le E_{x_k}\mathbf{l}_{\eta(B_{\ell,k}^1)}(x_k) \le E_{x_k}\mathbf{l}_{\eta(B_{\ell,k}^2)}(x_k) \le \widetilde{\mathbf{Y}}_k, \tag{4.6.33}$$

where $\widetilde{\mathbf{Y}}_k = \lim_{\ell\to\infty} E_{x_k}(\mathbf{l}_{\eta(B_{\sqrt{\theta_\ell}})}(x_k))$. Notice that $\widetilde{\mathbf{Y}}_k \stackrel{d}{=} \mathbf{Y}_1$ for all $\pi(k) \in \boldsymbol{\mathcal{T}}_\ell$. Also, since $\boldsymbol{\omega} \in (\mathcal{D}^2_\ell)^c$, the balls $B_{\eta(B_{\sqrt{\theta_\ell}})}(x_k)$ are disjoint for $\pi(k) \in \boldsymbol{\mathcal{T}}_\ell$ and therefore the $\widetilde{\mathbf{Y}}_k$'s are independent of each other. Furthermore, for the same reason, $\widetilde{\mathbf{Y}}_k$ is independent of $\boldsymbol{\gamma}_{\ell,l}$ for $l \neq k$ and $\pi(l) \in \boldsymbol{\mathcal{T}}_\ell$. It remains to control the dependence of $\widetilde{\mathbf{Y}}_k$ and $\boldsymbol{\gamma}_{\ell,k}$ for every $\pi(k) \in \boldsymbol{\mathcal{T}}_\ell$. This follows as in the proof of Lemma 4.1 in [40] (see the paragraph between (4.14)-(4.21)), where a variational formula, which is independent of x_k, is constructed for $E_{x_k}\mathbf{l}_{\eta(B_{\sqrt{\theta_\ell}}}(x_k)$. The proof of Lemma 4.19 is finished. □

We are now ready to prove that, $\mathbf{P}$-a.s., $\overline{\boldsymbol{\sigma}}_\ell \Rightarrow \boldsymbol{\sigma}$.

Lemma 4.20. $\mathbf{P}$*-a.s., for all* $\rho \geq 0$, $\lim_{\ell\to\infty} \mathbf{F}_\ell(\rho) = F_{\mathbf{Y},0}(\rho)$.

Proof. We establish the claim of Lemma 4.20 for fixed $\rho \geq 0$. Since $\mathbf{F}_\ell$ is a the distribution function and $F_{\mathbf{Y},0}$ is continuous in $\rho \geq 0$, this implies by Lemma 2.1 in [40] that the claim is true $\mathbf{P}$-a.s. for all $\rho \geq 0$. Let us first rewrite $\mathbf{F}_\ell$ and $F_{\mathbf{Y},0}$. For $\ell \in \mathbb{N}$ we define sequences of functions $f_{\ell,1} : (0,\infty) \times \mathbb{N} \to (0,\infty)$, $f_{\ell,2} : (0,\infty) \to (0,\infty)$ through

$$f_{\ell,1}(x,k) \equiv x \exp\left(-\rho/(\mathbf{Y}^1_k x^{1/b})\right) \mathbb{1}_{x > \bar{\epsilon}^b_\ell}, \qquad f_{\ell,2}(x) \equiv x \mathbb{1}_{x>\bar{\epsilon}^b_\ell}. \tag{4.6.34}$$

Then,

$$\mathbf{F}_\ell(\rho) = \left(\textstyle\sum_{1\leq k\leq \ell} f_{\ell,1}(\boldsymbol{\gamma}_{\ell,k}, k)\right)\left(\textstyle\sum_{1\leq k\leq\ell} f_{\ell,2}(\boldsymbol{\gamma}_{\ell,k})\right)^{-1}. \tag{4.6.35}$$

Moreover, setting $f_1 : (0,\infty)\times\mathbb{N} \to (0,\infty)$, $f_2 : (0,\infty) \to (0,\infty)$

$$f_1(x,k) \equiv x \exp\left(-\rho/(\mathbf{Y}^1_k x^{1/b})\right), \qquad f_2(x) \equiv x, \tag{4.6.36}$$

we have

$$F_{\mathbf{Y},0}(\rho) = \left(\textstyle\sum_{k\in\mathbb{N}} f_1(\boldsymbol{\gamma}_k, k)\right)\left(\textstyle\sum_{k\in\mathbb{N}} f_2(\boldsymbol{\gamma}_k)\right)^{-1}. \tag{4.6.37}$$

Thus, it suffices to establish that, $\mathbf{P}$-a.s.,

$$\lim_{\ell\to\infty} \textstyle\sum_{1\leq k\leq\ell} f_{\ell,1}(\boldsymbol{\gamma}_{\ell,k}, k) = \sum_{k=1}^\infty f_1(\boldsymbol{\gamma}_k, k), \tag{4.6.38}$$

$$\lim_{\ell\to\infty} \textstyle\sum_{1\leq k\leq\ell} f_{\ell,2}(\boldsymbol{\gamma}_{\ell,k}) = \sum_{k=1}^\infty f_2(\boldsymbol{\gamma}_k). \tag{4.6.39}$$

The proof of (4.6.38) and (4.6.39) relies on the fact that the collection $\{\boldsymbol{\gamma}_{\ell,k}, 1 \leq k \leq \ell\}$ and the points of the Poisson random measure $\{\boldsymbol{\gamma}_k, k \in \mathbb{N}\}$ are constructed from the same exponential random variables $\{E_i, i \in \mathbb{N}\}$. It follows the same idea of proof as the proof of Proposition 6.3 in [36] which in turn is based on the proof of Proposition 3.1 in [35]. Proposition 6.3 in [36] states that if f is continuous functions that obeys

$$\textstyle\int_{(0,\infty)} \min(f(x), 1) d\nu_{\alpha/b}(x) < \infty, \tag{4.6.40}$$

then, $\mathbf{P}$-a.s. $\lim_{\ell\to\infty} \sum_{1\leq k\leq\ell} f(\boldsymbol{\gamma}_{\ell,k}) = f(\boldsymbol{\gamma}_k)$. Since our functions are dependent on ℓ and not continuous, we cannot derive (4.6.38) and (4.6.39) from Proposition 6.3 in [36] directly but we can adapt its proof to our setting. For $\delta > 0$ we define the sets $I(\delta) \equiv \{k \in \mathbb{N} : \boldsymbol{\gamma}_k \geq \delta\}$ and $I^c(\delta) = \mathbb{N} \setminus I(\delta)$. As in the proof of Proposition 6.3 in [36], one can show that $\mathbf{P}$-a.s. the contribution coming from $k \in I^c(\delta)$ to f_2 vanishes, $\mathbf{P}$-a.s., as first $\ell \to \infty$ and then $\delta \to 0$. Since $f_{\ell,1} \leq f_1 \leq f_{\ell,2} \leq f_2$, it suffices to establish (4.6.38) and (4.6.39) for $k \in I(\delta)$. Let us first establish the claim of (4.6.39). We write

$$\textstyle\sum_{k\leq\ell, k\in I(\delta)} f_{\ell,2}(\boldsymbol{\gamma}_{\ell,k}) = \sum_{k\leq\ell,k\in I(\delta)} \boldsymbol{\gamma}_{n,k} - \sum_{k\leq\ell,k\in I(\delta)} \boldsymbol{\gamma}_{\ell,k} \mathbb{1}_{\boldsymbol{\gamma}_{\ell,k}\leq\bar{\epsilon}^b_\ell} \equiv (I) - (II). \tag{4.6.41}$$

From the proof of Proposition 3.1 in [36] we know that (I) tends, $\mathbf{P}$-a.s., as $\ell \to \infty$ and $\delta \to 0$, to $\sum_{k\in\mathbb{N}} f_2(\boldsymbol{\gamma}_k)$. It remains to establish that (II) vanishes, $\mathbf{P}$-a.s. Note that $(II) \leq \bar{\epsilon}_\ell^b |I(\delta)|$ and that $\mathbf{E}|I(\delta)| = \delta^{-\alpha/b}$. Thus, by a first order Chebyshev inequality,

$$\mathbf{P}\big(\textstyle\sum_{k\in I(\delta)} \gamma_{\ell,k} \mathbb{1}_{\gamma_{\ell,k}\leq \bar{\epsilon}_\ell^b} > \bar{\epsilon}_\ell^{b/2}\big) \leq \bar{\epsilon}_\ell^{b/2} \delta^{-\alpha/b}, \tag{4.6.42}$$

which tends to zero. Hence, (4.6.39) holds $\mathbf{P}$-a.s. Let $i=1$. It suffices to establish that, $\mathbf{P}$-a.s.,

$$\lim_{\ell\to\infty} \textstyle\sum_{k\leq \ell, k\in I(\delta)} f_{\ell,1}(\gamma_{\ell,k}) = \sum_{k\in I(\delta)} f_1(\boldsymbol{\gamma}_k). \tag{4.6.43}$$

We do this in two steps. First we establish that, $\mathbf{P}$-a.s.,

$$\lim_{\ell\to\infty} \big|\textstyle\sum_{k\in I(\delta)\cap \boldsymbol{\mathcal{T}}_\ell} \big(f_{\ell,1}(\gamma_{\ell,k}) - f_1(\boldsymbol{\gamma}_k)\big)\big| = 0. \tag{4.6.44}$$

In a second step we prove that, $\mathbf{P}$-a.s.,

$$\lim_{\ell\to\infty} \big(\textstyle\sum_{k\in I(\delta), k>\ell} \boldsymbol{\gamma}_k + \sum_{k\leq \ell, k\in I(\delta)} \boldsymbol{\gamma}_k \mathbb{1}_{\gamma_{\ell,k}\leq \epsilon_n}\big) = 0. \tag{4.6.45}$$

Since $f_{\ell,1} = 0$ for $k \notin \boldsymbol{\mathcal{T}}_\ell$, (4.6.44) and (4.6.45) imply (4.6.43). Let us first establish (4.6.45). By definition of $\boldsymbol{\gamma}_k$ (see (4.6.17)),

$$\textstyle\sum_{k>\ell, k\in I(\delta)} \boldsymbol{\gamma}_k \leq \sum_{k>\ell} \big(\sum_{i=1}^k E_i\big)^{-b/\alpha} = \sum_{k>\ell} k^{-b/\alpha} \big(\frac{1}{k}\sum_{i=1}^k E_i\big)^{-b/\alpha}. \tag{4.6.46}$$

By the law of large numbers, we can bound, $\mathbf{P}$-a.s., for $k \geq k_0$, $1/k\sum_{i=1}^k E_i \geq 1-\varepsilon$. Since $b > \alpha$, we know that (4.6.46), and hence also the first sum in (4.6.45), tend $\mathbf{P}$-a.s. to zero. Let us now bound the second sum in (4.6.45). Using the definition of $\gamma_{\ell,k}$ and $\boldsymbol{\gamma}_k$ (see (4.5.2) and (4.6.17)), the fact that G^{-1} is regularly varying with index $-1/\alpha$ at $0+$ and the law of large numbers one can show that, $\mathbf{P}$-a.s., $\sup_{k\in B_1} |\gamma_{\ell,k}/\boldsymbol{\gamma}_k - 1| \to 0$. Thus,

$$\textstyle\sum_{k\in B_1} \boldsymbol{\gamma}_k \mathbb{1}_{\gamma_{\ell,k}\leq \bar{\epsilon}_\ell} \leq \bar{\epsilon}_\ell \sup_{k\in I(\delta)} |\boldsymbol{\gamma}_k/\gamma_{\ell,k} - 1||I(\delta)|, \tag{4.6.47}$$

which vanishes $\mathbf{P}$-a.s. by (4.6.42). Therefore, the claim of (4.6.45) holds true. Let $k \in I(\delta)\cap \boldsymbol{\mathcal{T}}_\ell$. Set $X_{\ell,k} \equiv \gamma_{\ell,k}^{1/b}\mathbf{Y}_k$ and $X_k \equiv \boldsymbol{\gamma}_k^{1/b}\mathbf{Y}_k$. Suppose without loss of generality that $X_{\ell,k} \leq X_k$. Using for $x \geq 0$ the inequalities $1 - e^{-x} \leq x$ and $xe^{-x} \leq 1$, we get that

$$\begin{aligned} |f_{\ell,1}(\gamma_{\ell,k}) - f_1(\boldsymbol{\gamma}_k)| &\leq |\gamma_{\ell,k} - \boldsymbol{\gamma}_k| e^{-\rho/X_{\ell,k}} + \boldsymbol{\gamma}_k e^{-\rho/X_k} |1 - \exp(-\rho/X_{\ell,k} + \rho/X_k)| \\ &\leq |\gamma_{\ell,k} - \boldsymbol{\gamma}_k| + \boldsymbol{\gamma}_k e^{-\rho/X_k} \rho/X_k |1 - X_k/X_{\ell,k}| \\ &\leq \boldsymbol{\gamma}_k \big(|1 - \gamma_{\ell,k}/\boldsymbol{\gamma}_k| + |1 - (\boldsymbol{\gamma}_k/\gamma_{\ell,k})^{1/b}|\big). \end{aligned} \tag{4.6.48}$$

When $X_{\ell,k} > X_k$ we can proceed similarly. Thus,

$$\big|\textstyle\sum_{k\in I(\delta)\cap \boldsymbol{\mathcal{T}}_\ell} \big(f_{\ell,1}(\gamma_{\ell,k}) - f_1(\boldsymbol{\gamma}_k)\big)\big| \leq \sup_{k\in I(\delta)\cap \boldsymbol{\mathcal{T}}_\ell} C\big(|1 - \frac{\gamma_{\ell,k}}{\boldsymbol{\gamma}_k}|\big) \sum_{k=1}^\infty \boldsymbol{\gamma}_k. \tag{4.6.49}$$

Now, by Proposition 6.3 in [37], $\mathbf{P}$-a.s., $\sum_k \boldsymbol{\gamma}_k < \infty$. As remarked above (4.6.47), $\mathbf{P}$-a.s., $\sup_{k\in B_1} |\gamma_{\ell,k}/\boldsymbol{\gamma}_k - 1| \to 1$. Hence, (4.6.49) tends to zero. This finishes the proof of (4.6.44) and together with (4.6.45) the proof of Lemma 4.20.

□

Proof of Proposition 4.17. We are now ready to conclude the proof of Proposition 4.17. First, by Lemma 4.18, we know that, $\mathbf{P}$-a.s., $|\boldsymbol{\sigma}_\ell - \overline{\boldsymbol{\sigma}}_\ell|$ converges in $\mathcal{P}_{\boldsymbol{\mu}_{A,b}}$-probability to zero. Then, Lemma 4.19 and Lemma 4.20 establish that, $\mathbf{P}$-a.s., $\overline{\boldsymbol{\sigma}}_\ell \Rightarrow \boldsymbol{\sigma}$. By Slutzky's Theorem we therefore obtain that, $\mathbf{P}$-a.s., $\boldsymbol{\sigma}_\ell \Rightarrow \boldsymbol{\sigma}$. □

4.6.2 Convergence of the initial jump on short time scales

In this section we study the initial jump when we observe the process on short time scales as defined in (4.1.32). In order to make this statement precise, we define the rescaled initial jump is as

$$\sigma_\ell^{sh} \equiv (c_\ell^{sh})^{-1} e_1 \sum_{x\in A} (\tau(x))^{1-\theta} \left(\sum_{y\sim x} (\tau(y))^\theta\right)^{-1} \mathbb{1}_{J(0)=x}, \tag{4.6.50}$$

and its representation on $(\mathbf{\Omega}, \boldsymbol{\mathcal{F}}, \mathbf{P})$ through

$$\boldsymbol{\sigma}_\ell^{sh} \equiv (c_\ell^{sh})^{-1} e_1 \sum_{k=1}^{\ell} \ell^{1/\alpha} \gamma_{\pi(k),\ell}^{(1-\theta)/b} \left(\sum_{y\sim x_k} (\tau(y))^\theta\right)^{-1} \mathbb{1}_{J(0)=x_k}. \tag{4.6.51}$$

Let us now establish the following proposition.

Proposition 4.21. *Let c_ℓ^{sh} be given by* (4.1.18). *Let $\mathbf{Y}$ be a collection of i.i.d. random variables on $(\mathbf{\Omega}, \boldsymbol{\mathcal{F}}, \mathbf{P})$ whose elements are as in* (4.1.31). *Let $\mathbf{\Gamma}$ be the Poisson random measure defined in* (4.6.17). *Then, $\mathbf{P}$-a.s., $\boldsymbol{\sigma}_\ell^{sh} \Rightarrow \boldsymbol{\sigma}^{sh}$, where $\boldsymbol{\sigma}^{sh}$ has distribution function $\mathbf{P}(\boldsymbol{\sigma} \le u) = F_{\mathbf{Y},\theta}(u t_w^{-(1-\theta)\delta'})$, $u > 0$, where $F_{\mathbf{Y},\theta}$ is given by* (4.1.17) *and where $\delta' = \delta/(1+\alpha)\mathbb{1}_{d=1} + \delta\mathbb{1}_{d\ge 2}$.*

Proof. By Lemma 4.16 and by (4.6.13), we know for all $\varepsilon > 0$, $\mathbf{P}$-a.s., for ℓ large enough

$$|\mathcal{P}_{\mu_{A,b}}(\boldsymbol{\sigma}_\ell^{sh} > u) - \mathcal{P}_{\mu_{A,b}}(\boldsymbol{\sigma}_\ell^{sh} \mathbb{1}_{\mathcal{D}_\ell^c} > u | J(0) \in T)| < \varepsilon, \qquad \forall u > 0, \tag{4.6.52}$$

where $\mathcal{D}_\ell = \mathcal{D}_\ell^1 \cup \mathcal{D}_\ell^2$ is as in (4.6.9) and (4.6.10). By definition of $\mathcal{D}_\ell$, the balls $\{B_{\overline{\theta}_\ell}(x_k), \pi(k) \in \boldsymbol{\mathcal{T}}_\ell\}$ are disjoint and so $\sum_{y\sim x_k} (\tau(y))^\theta$ is a collection of i.i.d random variables which is independent of $\{\gamma_{\ell,k}, k \in \boldsymbol{\mathcal{T}}_\ell\}$. Hence, writing $\mathbf{Y}_i \equiv (\sum_{k=1}^{2d} (\tau_{i,k})^\theta)^{-1}$, we get for $u > 0$,

$$\begin{aligned}\mathcal{P}_{\boldsymbol{\mu}_{A,b}}(\boldsymbol{\sigma}_\ell^{sh} > u) &= \sum_{k\in\boldsymbol{\mathcal{T}}_\ell} \gamma_{\ell,k} \left(\sum_{k\in\boldsymbol{\mathcal{T}}_\ell} \gamma_{\ell,k}\right)^{-1} P\left(e_1 (\gamma_{\ell,k})^{(1-\theta)/b} \mathbf{Y}_k > u c_\ell^{sh} \ell^{-(1-\theta)/\alpha}\right) \\ &= \sum_{k\in\boldsymbol{\mathcal{T}}_\ell} \gamma_{\ell,k} \left(\sum_{k\in\boldsymbol{\mathcal{T}}_\ell} \gamma_{\ell,k}\right)^{-1} \exp\left(-u/\{t_w^{(1-\theta)\delta'} \gamma_{\ell,k}^{(1-\theta)/b} \mathbf{Y}_k\}\right),\end{aligned} \tag{4.6.53}$$

where we used the definition of c_ℓ^{sh} in the last step. As in the proof of Lemma 4.20 one can show that, $\mathbf{P}$-a.s., for all $u > 0$,

$$\lim_{\ell\to\infty} \mathcal{P}_{\boldsymbol{\mu}_{A,b}}(\overline{\boldsymbol{\sigma}}_\ell^{sh} > u) = \sum_{i=1}^{\infty} \frac{\gamma_i}{\sum_{i=1}^{\infty} \gamma_i} \exp\left(-u/\left(t_w^{(1-\theta)\delta'} \gamma_i^{(1-\theta)/b} \mathbf{Y}_i\right)\right), \tag{4.6.54}$$

as desired. The proof of Proposition 4.21 is finished. □

4.7 Conclusions of the proofs of the convergence results of Section 4.1.1 when $\mu = \mu_{A,b}$

In this sections we conclude the proofs of Theorems 4.2, 4.4, 4.6, 4.7, and Corollary 4.3 in the following order: Section 4.7.1 is devoted to Theorem 4.2, Section 4.7.2 to Theorem 4.6, Section 4.7.3 to Corollary 4.3, Section 4.7.4 to Theorem 4.4, , and Section 4.7.5 to Theorem 4.7

4.7.1 Conclusion of the proof of Theorem 4.2

We now derive the claim of Theorem 4.2 from Theorem 4.1, Proposition 4.17, and Lemma 4.15. Let us first establish that the conditions of Theorem 4.1 are met for $\boldsymbol{\sigma}_\ell$ and $\mathbf{S}_\ell^b$ as in (4.5.8) and (4.5.9). By Proposition 4.17, we know that $\mathbf{P}$-a.s., $\boldsymbol{\sigma}_\ell \Rightarrow \boldsymbol{\sigma}$. Also, c_ℓ^{int} satisfies Condition (A-2) and therefore, writing $\mathbf{R}(c_\ell^{int} t_w, c_\ell^{int} t)$ for the representation of $R(c_\ell^{int} t_w, c_\ell^{int} t)$ on $(\mathbf{\Omega}, \boldsymbol{\mathcal{F}}, \mathbf{P})$, we know that, $\mathbf{P}$-a.s., for all $\delta > 0$,

$$\lim_{\ell\to\infty} \mathbf{R}(c_\ell^{int} t_w, c_\ell^{int} t) = \boldsymbol{\mathcal{C}}_{\infty,\delta}(t_w, t), \tag{4.7.1}$$

where $\boldsymbol{\mathcal{C}}_{\infty,\delta}$ is as in (4.1.21) when we substitute $F_{\mathbf{Y},0}$ for $F_{Y,0}$. To conclude the proof, it remains to apply Lemma 4.15 to get results on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\{X^1_{\ell,k}\} = \{\gamma_{\ell,k}\}$, $\{\hat{X}^1_{\ell,k}\} = \{\boldsymbol{\gamma}_{\ell,k}\}$, $\{X^2_{\ell,k}\} = \{\tau(x), x \notin A\}$ and $g_\ell(\{X^1_{\ell,k}\}, \{X^2_{\ell,k}\}) = R(c^{int}_\ell t_w, c^{int}_\ell t)$. Combining (4.7.1) and Lemma 4.15 yields, in $\mathbb{P}$-law,

$$\lim_{\ell\to\infty} R(c^{int}_\ell t_w, c^{int}_\ell t) = \mathcal{C}_{\infty,\delta}(t_w, t). \tag{4.7.2}$$

This finishes the proof of Theorem 4.2.

4.7.2 Conclusion of the proof of 4.6

In this section we determine the limit of $\Pi(c^{sh}_\ell t_w, c^{sh}_\ell t)$ where c^{sh}_ℓ is as in (4.1.18). Let $t_w, t > 0$ and fix $\delta > 0$. Recall that $\Pi(c^{sh}_\ell t_w, c^{sh}_\ell t)$ is the probability that X stays in the same point during the time interval $(c^{sh}_\ell t_w, c^{sh}_\ell (t_w + t))$. This event can be decomposed into two events. The first is that $X(c^{sh}_\ell t') = X(0)$ for all $t' \le t_w + t$ and the second is that there is $t' \le t_w$ such that $X(0) \neq X(c^{sh}_\ell t')$ and $X(c^{sh}_\ell t_w) = X(c^{sh}_\ell (t_w + t'))$ for all $t' \le t$. Using the definition of σ^{sh}_ℓ in (4.6.50), we thus have

$$\begin{aligned}\Pi(c^{sh}_\ell t_w, c^{sh}_\ell t) &= \mathcal{P}_{\mu_{A,b}}(\sigma^{sh}_\ell \ge t + t_w) \\ &+ \mathcal{P}_{\mu_{A,b}}(\sigma^{sh}_\ell < t_w, X(c^{sh}_\ell t_w) = X(c^{sh}_\ell (t_w + s), \ \forall 0 \le s \le t).\end{aligned} \tag{4.7.3}$$

In Proposition 4.21 we have established that, $\mathbf{P}$-a.s., $\boldsymbol{\sigma}^{sh}_\ell \Rightarrow \boldsymbol{\sigma}^{sh}$. We use this now to determine the limit as $\ell \to \infty$ and $t_w \to \infty$ such that $t/t_w^{(1-\theta)\delta'} = \rho$, where $\delta' = \delta/(1+\alpha)\mathbb{1}_{d=1} + \delta\mathbb{1}_{d\ge 2}$ of (4.7.3). Let $\mathbf{\Pi}(c^{sh}_\ell t_w, c^{sh}_\ell t)$ denote the probability of no jump in the representation on $(\mathbf{\Omega}, \boldsymbol{\mathcal{F}}, \mathbf{P})$. Eq. (4.7.3) also holds in this representation. Let us now compute the limits of the summands in (4.7.3). By Proposition 4.21, we know that, $\mathbf{P}$-a.s.,

$$\lim\nolimits_{\ell\to\infty} \mathcal{P}_{\boldsymbol{\mu}_{A,b}}(\boldsymbol{\sigma}^{sh}_\ell > t + t_w) = 1 - F_{\mathbf{Y},\theta}\big((t+t_w)t_w^{-(1-\theta)\delta'}\big), \tag{4.7.4}$$

where $F_{\mathbf{Y},\theta}$ is as in Proposition 4.21. Now,

$$\lim\nolimits_{t_w\to\infty:\ t/t_w^{(1-\theta)\delta}=\rho} \big(1 - F_{\mathbf{Y},\theta}\big((t+t_w)t_w^{-(1-\theta)\delta'}\big)\big) = 1 - F_{\mathbf{Y},\theta}(\rho), \tag{4.7.5}$$

where we used the assumption $(1-\theta)\delta' > 1$. Also, by (4.7.4)

$$\mathcal{P}_{\boldsymbol{\mu}_{A,b}}(\boldsymbol{\sigma}^{sh}_\ell \le t_w) \overset{\ell\to\infty}{\longrightarrow} F_{\mathbf{Y},\theta}\big(t_w^{1-(1-\theta)\delta'}\big) \overset{t_w\to\infty}{\longrightarrow} 0. \tag{4.7.6}$$

The left hand side of (4.7.6) is an upper bound for the second summand in (4.7.3). Thus, plugging (4.7.5) and (4.7.6) into (4.7.3), we obtain, $\mathbf{P}$-a.s.,

$$\lim\nolimits_{t_w\to\infty:\ t/t_w^{(1-\theta)\delta}=\rho} \lim\nolimits_{\ell\to\infty} \mathbf{\Pi}(c^{sh}_\ell t_w, c^{sh}_\ell t) = 1 - F_{\mathbf{Y},\theta}(\rho). \tag{4.7.7}$$

It remains to derive a statement in the original probability space. We apply Lemma 4.15 to $\{X^1_{\ell,k}\} = \{\gamma_{\ell,k}\}$, $\{\hat{X}^1_{\ell,k}\} = \{\boldsymbol{\gamma}_{\ell,k}\}$, $\{X^2_{\ell,k}\} = \{\tau(x), x \notin A\}$ and $g_\ell(\{X^1_{\ell,k}\}, \{X^2_{\ell,k}\}) = \Pi(c^{sh}_\ell t_w, c^{sh}_\ell t)$. Combining (4.7.8) and Lemma 4.15 yields, in $\mathbb{P}$-law,

$$\lim\nolimits_{t_w\to\infty:\ t/t_w^{(1-\theta)\delta}=\rho} \lim\nolimits_{\ell\to\infty} \Pi(c^{sh}_\ell t_w, c^{sh}_\ell t) = 1 - F_{Y,\theta}(\rho). \tag{4.7.8}$$

This finishes the proof of Theorem 4.6.

4.7.3 Conclusion of the proof of Corollary 4.3

We study the behavior of $\mathcal{C}_{\infty,\delta}(t_w,t)$, as $t_w \to \infty$ and $t/t_w^\delta = \rho$ for $\rho \geq 0$, in this section. For this, we distinguish three cases with respect to δ. First, let $\delta < 1$. Since $F_{Y,0}$ is a distribution function,

$$\lim_{\substack{t_w\to\infty \\ t/t_w^\delta=\rho}} \left(1 - F_{Y,0}\left(\tfrac{t_w+t}{t_w^\delta}\right)\right) = \lim_{\substack{t_w\to\infty \\ t/t_w^\delta=\rho}} \left(1 - F_{Y,0}\left(t_w^{1-\delta}+\rho\right)\right) = 0. \tag{4.7.9}$$

Also, since $v \mapsto \mathrm{Asl}_\alpha\left(\frac{t_w-v}{t_w-v+t}\right)$ is decreasing, we have for $\varepsilon \in (0, 1-\delta)$

$$\begin{aligned} &\textstyle\int_0^{t_w} \mathrm{Asl}_\alpha\left(\frac{t_w-v}{t_w-v+t}\right) dF_{Y,0}\left(\frac{v}{t_w^\delta}\right) \\ &\geq \mathrm{Asl}_\alpha\left(1/\{1-t/(t_w-t_w^{\delta+\varepsilon})\}\right)\left[F_{Y,0}(t_w^\varepsilon) - F_{Y,0}(0)\right]. \end{aligned} \tag{4.7.10}$$

The term $F_{Y,0}(t_w^\varepsilon) - F_{Y,0}(0)$ tends to 1 as $t_w \to \infty$ and the second term on the right hand side of (4.7.10) tends to one as $t_w \to \infty$ and $t/t_w^\delta = \rho$. Thus, the proof for $\delta < 1$ is complete.

Let $\delta = 1$ and take t_w, t such that $t/t_w^\delta = \rho$. Then, $(t_w+t)t_w^{-\delta} = 1+\rho$ and the first term in $\mathcal{C}_{\infty,\delta}(t_w,t)$ is equal to $1 - F_{Y,0}(1+\rho)$. Moreover, substituting $u = vt_w$ yields

$$\textstyle\int_0^{t_w} \mathrm{Asl}_\alpha\left(\frac{t_w-v}{t_w-v+t}\right) dF_{Y,0}(vt_w^{-1}) = \int_0^1 \mathrm{Asl}_\alpha\left(\frac{1-w}{1-w+\rho}\right) dF_{Y,0}(w), \tag{4.7.11}$$

which is as desired. This finishes the proof for $\delta = 1$.

Finally, let $\delta > 1$. Since Asl_α is a probability measure,

$$1 - F_{Y,0}((t_w+t)t_w^{-\delta}) \leq \mathcal{C}_{\infty,\delta}(t_w,t) \leq 1 - F_{Y,0}((t_w+t)t_w^{-\delta}) + F_{Y,0}(t_w^{1-\delta}). \tag{4.7.12}$$

The last term in (4.7.12) tends for $t_w \to \infty$ to zero. Moreover, $1 - F_{Y,0}(t_w^{1-\delta} + tt_w^{-\delta})$ tends for $t_w \to \infty$ such that $t/t_w^\delta = \rho$ to $1 - F_{Y,0}(\rho)$. The proof for $\delta > 1$ is finished.

4.7.4 Conclusion of the proof of 4.4

In this section we determine the limit of $R(c_\ell^{lo}t_w, c_\ell^{lo}t)$, as $c_\ell^{lo} \to \infty$. Let $a_\ell^{lo}, \theta_\ell^{lo}$ as in (4.1.14). We establish now that the assumptions of Theorem 4.1 are verified in this setting. Note first that $|A| = \ell \ll c_\ell^{lo}$ and so Condition (A-2) is satisfied. It remains to establish that Condition (A-1) holds for $\sigma = 0$. By (4.2.28) it suffices to prove in this respect that the set $\mathcal{T}_\ell = \{x \in A : \tau(x) > c_\ell^{lo}\epsilon_\ell^{lo}\}$, where ϵ_ℓ^{lo} is as in (4.6.1), is $\mathbb{P}$-a.s. empty. We have that, $\mathbb{P}(\mathcal{T}_\ell \neq \emptyset) \leq \ell^{-m+1}(\epsilon_\ell^{lo})^{-\alpha}$, which vanishes. Thus, $\mathbb{P}$-a.s., $\mathcal{T}_\ell = \emptyset$ and so $\sigma_\ell \Rightarrow 0$ by (4.2.28).

4.7.5 Conclusion of the proof of Theorem 4.7

This section contains the proof of Theorem 4.7. For this, we verify the conditions of Theorem 4.1. By (4.1.34) and (4.1.36), Condition (A-2) is satisfied and it remains to check Condition (A-1). Since k is a finite number, nothing changes in the calculations of the proofs of Theorem 4.2 and Theorem 4.4 when $A = \bigcup_{i=1}^k A_i$. What remains to be checked is the mutual independence of $\{\Gamma^i\}_{i=1}^k$, the mutual independence of the collection $\{Y^i\}_{i=1}^k$, and the independence of $\{\Gamma^i\}_{i=1}^k$, and $\{Y^i\}_{i=1}^k$. By construction, $\{\{\tau(x), x \in A_i\}, i = 1, \ldots, k\}$ is a family of mutually independent collections, which implies that the Γ^i's are mutually independent. We define the events $\mathcal{D}_n^1$ and $\mathcal{D}_n^2$ as in (4.6.9) and (4.6.10), with the set $\mathcal{T}_\ell$ given by

$$\mathcal{T}_\ell \equiv \textstyle\bigcup_{i=1}^k \mathcal{T}_{\ell,m_i}, \tag{4.7.13}$$

where $\mathcal{T}_{\ell,m_i} \equiv \{x \in A : \tau(x) > \ell^{m_i/\alpha}\epsilon_\ell\}$. Then, the balls with radius $\bar{\theta}_\ell$ around points x_k for $\pi(k) \in \mathcal{T}_\ell$ are disjoint which implies that the $\bar{Y}^i$'s are mutually independent, and moreover that $\{\Gamma^i\}_{i=1}^k$ is independent of $\{Y^i\}_{i=1}^k$. The proof of Theorem 4.7 is complete.

4.8 Properties of $F_{Y,\theta}$

In this section we study the behavior of the distribution function of $F_{Y,\theta}$, $\theta \in [0,1)$. We establish in Proposition 4.22 that its expectation can be written as a Laplace transform of a random variable. This proposition is taken from [38]. The proof of assertions (4.1.27)-(4.1.30) of Theorem 4.5 follows from Proposition 4.22, dominated convergence, and simple but lengthy calculations.

Let us now write $F_{Y,\theta}$, respectively $\overline{F}_{Y,\theta} \equiv 1 - F_{Y,\theta}$ as a Laplace transform of a random variable Z having density function given by

$$df_{\alpha/b}(z) \equiv (\alpha/b)\, z^{-\alpha/b} dz \int_0^\infty d\lambda \exp\{-(\Gamma(1-\alpha/b)\lambda^{\alpha/b} + \lambda z)\}\,, \quad z \geq 0\,. \tag{4.8.1}$$

Proposition 4.22. *Denote by G the distribution function of* Y_1. *For* $\rho \geq 0$ *and* $\theta \in [0,1)$,

$$\mathbb{E}\overline{F}_{Y,\theta}(\rho) = \int_0^\infty dG(y) \int_0^\infty df_{\alpha/b}(z) \exp\{-\rho z^{-(1-\theta)/b} y^{-1}\}. \tag{4.8.2}$$

Proof of Proposition 4.22. Fix $\rho \geq 0$ and $\theta \in [0,1)$. The collection Y is independent of Γ. Therefore, conditioning on Γ and using the fact that the Y_j's are i.i.d.,

$$\mathbb{E}(\overline{F}_{Y,\theta}(\rho)|\Gamma) = \textstyle\sum_{j=1}^\infty (\sum_{j=1}^\infty \gamma_j)^{-1} \gamma_j\, \mathbb{E}(\exp\{-\rho\gamma_j^{-(1-\theta)/b}(Y_1)^{-1}\}|\Gamma). \tag{4.8.3}$$

Thus, each summand is multiplied with Y_1 and we obtain by Fubini

$$\mathbb{E}\overline{F}_{Y,\theta}(\rho) = \int_0^\infty dG(y)\mathbb{E}(\textstyle\sum_{j=1}^\infty \gamma_j\ \exp\{-\rho/(\gamma_j^{(1-\theta)/b} y)^{-1}\}(\sum_{j=1}^\infty \gamma_j)^{-1}). \tag{4.8.4}$$

Hence it suffices to calculate for fixed $y \in (0,\infty)$,

$$\mathbb{E}\overline{F}_{Y,\theta}(\rho) = \mathbb{E}(\textstyle\sum_{j=1}^\infty \gamma_j\ \exp\{-\rho/y\gamma_j^{-(1-\theta)/b}\}(\sum_{j=1}^\infty \gamma_j)^{-1}). \tag{4.8.5}$$

Let $y \in (0,\infty)$ be fixed and set $\rho' \equiv \rho y$. Given an integer $m > 0$ set

$$\sigma_m^+(\rho') = \textstyle\sum_{i=1}^\infty \gamma_i \exp\{-\rho'/\gamma_i^{(1-\theta)/b}\}\mathbb{1}_{\{\gamma_i > 1/m\}}, \tag{4.8.6}$$

$$\sigma_m^-(\rho') = \textstyle\sum_{i=1}^\infty \gamma_i \exp\{-\rho'/\gamma_i^{(1-\theta)/b}\}\mathbb{1}_{\{\gamma_i \leq 1/m\}}, \tag{4.8.7}$$

and write $\mathbb{E}\overline{F}_{Y,\theta}(\rho') \equiv h_m^+(\rho') + h_m^-(\rho')$ where

$$h_m^+(\rho') = \mathbb{E}\frac{\sigma_m^+(\rho')}{\sigma_m^+(0)+\sigma_m^-(0)}, \quad h_m^-(\rho') = \mathbb{E}\frac{\sigma_m^-(\rho')}{\sigma_m^+(0)+\sigma_m^-(0)}. \tag{4.8.8}$$

Clearly $\mathbb{E}\overline{F}_{Y,\theta}(\rho') = \lim_{m\to\infty}(h_m^+(\rho') + h_m^-(\rho'))$. Observing that

$$h_m^-(\rho') \leq \mathbb{E}\frac{\sigma_m^-(0)}{\sigma_m^+(0)+\sigma_m^-(0)} e^{-\rho' m^{(1-\theta)/b}} \leq e^{-\rho' m^{(1-\theta)/b}}, \tag{4.8.9}$$

we have $\lim_{m\to\infty} h_m^-(\rho') = 0$. It thus suffices to establish that

$$\lim_{m\to\infty} h_m^+(\rho') = \int_0^\infty df_{\alpha/b}(z) \exp\{-\rho'/z^{(1-\theta)/b}\}. \tag{4.8.10}$$

To this end let $\{\gamma_i^-, i \in \mathbb{N}\}$ and $\{\gamma_i^+, i \in \mathbb{N}\}$ be, respectively, the marks of $\mathrm{PRM}(\nu_{\alpha/b,m}^-)$ and $\mathrm{PRM}(\nu_{\alpha/b,m}^+)$ where, for a Borel set $A \subseteq (0,\infty)$, $\nu_{\alpha/b,m}^-(A) = \mu(A \cap (0,1/m))$ and $\nu_{\alpha/b,m}^+(A) = \mu(A \cap [1/m,\infty))$. Namely, these are the restrictions of $\mathrm{PRM}(\mu)$ to the sets $(0,1/m)$

and $[1/m, \infty)$ respectively. Since these sets are disjoint, $\mathrm{PRM}(\nu^-_{\alpha/b,m})$ and $\mathrm{PRM}(\nu^+_{\alpha/b,m})$ are independent. A well known necessary and sufficient condition for the sums $\sigma^\pm_m(\rho')$ in (4.8.6)-(4.8.7) to be absolutely convergent with probability one is that (see e.g. [46], Campbell's Theorem)

$$\int d\nu^\pm_{\alpha/b,m}(x) \min\left(xe^{-\rho'/x^{(1-\theta)/b}}, 1\right) < \infty. \tag{4.8.11}$$

But clearly, (4.8.11) holds true for all $m \geq 0$, all $\rho' \geq 0$, and all $\alpha \leq b \leq 1$, $\alpha/b < 1$. Thus, using the identity $\frac{1}{X} = \int_0^\infty d\lambda e^{-\lambda X}$ and Fubini, we can write, with obvious notation,

$$\begin{aligned} h^+_m(\rho') &= \mathbb{E} \int_0^\infty d\lambda \, \sigma^+_m(\rho') e^{-\lambda\{\sigma^+_m(0)+\sigma^-_m(0)\}}, \\ &= \int_0^\infty d\lambda \, \mathbb{E}^+ \left(\sigma^+_m(\rho') e^{-\lambda\sigma^+_m(0)}\right) \mathbb{E}^- \left(e^{-\lambda\sigma^-_m(0)}\right). \end{aligned} \tag{4.8.12}$$

Lemma 4.23. *For $\lambda \geq 0$ set*

$$\psi^\pm_m(\lambda) = \int_0^\infty d\nu^\pm_{\alpha/b,m}(x) \left(1 - e^{-\lambda x}\right). \tag{4.8.13}$$

Then

$$\mathbb{E}^- \left(e^{-\lambda\sigma^-_m(0)}\right) = e^{-\psi^-_m(\lambda)}, \tag{4.8.14}$$

$$\mathbb{E}^+ \left(\sigma^+_m(\rho') e^{-\lambda\sigma^+_m(0)}\right) = \int_0^\infty d\nu^+_{\alpha/b,m}(x) \, x e^{-\left(\rho' x^{-(1-\theta)/b} + \lambda x\right)} e^{-\psi^+_m(\lambda)}. \tag{4.8.15}$$

and both expectations are finite for all $m \geq 0$, all $\rho' \geq 0$, and all $\alpha < b \leq 1$, $\alpha/b < 1$.

Proof of Lemma 4.23. The expectation (4.8.14) is bounded by 1 since $\sigma^-_m(0) \geq 0$. The boundedness of (4.8.15) follows from (4.8.11). We are left to evaluate the expectations appearing in (4.8.12). Eq. (4.8.14) is the characteristic functional of $\mathrm{PRM}(\nu^-_{\alpha/b,m})$. We thus only have to prove (4.8.15). Observe that for $m > 0$, $\mathrm{PRM}(\nu^+_{\alpha/b,m})$ has finite intensity measure. Then, as is well known (see e.g. Chapter 2.4 in [46]), conditioning on the total number of points of the process turns it into a Bernoulli process. More precisely, if N denotes the counting function of $\mathrm{PRM}(\mu^+_\varepsilon)$ then, conditional on $N([\varepsilon, \infty)) = n$, the points $X_1, \ldots, X_n$ of $\mathrm{PRM}(\nu^+_{\alpha/b,m})$ are i.i.d. random variables with common distribution

$$\nu^+_{\alpha/b,m}(dx)\{\nu_{\alpha/b}([1/m, \infty))\}^{-1}. \tag{4.8.16}$$

Thus we can write

$$\mathbb{E}^+ \left(\sigma^+_m(\rho') e^{-\lambda\sigma^+_m(0)}\right) = \sum_{n=0}^\infty (\nu_{\alpha/b}([1/m, \infty)))^n \frac{e^{-\nu_{\alpha/b}([1/m,\infty))}}{n!} \mathbb{E}^+_n \left(\sigma^+_m(\rho') e^{-\lambda\sigma^+_m(0)}\right), \tag{4.8.17}$$

where $\mathbb{E}^+_n \left(\sigma^+_m(\rho') e^{-\lambda\sigma^+_m(0)}\right) \equiv \mathbb{E}^+ \left(\sigma^+_m(\rho') e^{-\lambda\sigma^+_m(0)} \mid N([1/m, \infty)) = n\right)$. Now,

$$\begin{aligned} \mathbb{E}^+_n \left(\sigma^+_m(\rho') e^{-\lambda\sigma^+_m(0)}\right) &= \mathbb{E}^+_n \sum_{i=1}^n X_i e^{-\rho'/X_i^{(1-\theta)/b}} \exp\left\{-\lambda \sum_{j=1}^n X_j\right\} \\ &= \sum_{i=1}^n \mathbb{E}^+_n \left(X_i e^{-\rho'/X_i^{(1-\theta)/b} - \lambda X_i}\right) \left(\mathbb{E}^+_n \exp\left\{-\lambda \sum_{\substack{j=1 \\ j\neq i}}^n X_j\right\}\right) \\ &= nAB^{n-1} \end{aligned} \tag{4.8.18}$$

where the one before last line follows by independence, and where in the last line

$$A \equiv \int xe^{-\rho'/x^{(1-\theta)/b} - \lambda x} \left(\nu^+_{\alpha/b,m}(dx)/\nu_{\alpha/b}([1/m, \infty))\right), \tag{4.8.19}$$

$$B \equiv \int e^{-\lambda x} \left(\nu^+_{\alpha/b,m}(dx)/\nu_{\alpha/b}([1/m, \infty))\right). \tag{4.8.20}$$

Combining (4.8.17) and (4.8.18) we obtain

$$\mathbb{E}^+\left(\sigma_m^+(\rho')e^{-\lambda\sigma_m^+(0)}\right) = A\nu_{\alpha/b}([1/m,\infty))e^{-\nu_{\alpha/b}([1/m,\infty))(1-B)}, \tag{4.8.21}$$

and inserting (4.8.19) and (4.8.20) into (4.8.21) yields (4.8.14). Finally, the boundedness of (4.8.14) is immediate whereas that of (4.8.15) follows from explicit calculations. $\square$

Inserting (4.8.14) and (4.8.15) into (4.8.12) yields

$$h_m^+(\rho') = \int_0^\infty d\lambda \int_0^\infty \nu_{\alpha/b,m}^+(dx)\, xe^{-\left(\rho'/x^{(1-\theta)/b}+\lambda x\right)}e^{-\{\psi_m^+(\lambda)+\psi_m^-(\lambda)\}}. \tag{4.8.22}$$

By (4.8.13) and the definition of $\nu_{\alpha/b,m}^\pm$, $\psi_m^+(\lambda)+\psi_m^-(\lambda) = \int_0^\infty d\nu_{\alpha/b,m}(x)\left(1-e^{-\lambda x}\right) = \Gamma(1-\alpha/b)\lambda^{\alpha/b}$. Together with this and the definition of $\nu_{\alpha/b}$, (4.8.22) becomes

$$h_m^+(\rho') = \int_0^\infty d\lambda \int_{1/m}^\infty dx\, (\alpha/b)\, x^{-\alpha/b}e^{-\rho'/x^{(1-\theta)/b}}e^{-(\Gamma(1-\alpha/b)\lambda^{\alpha/b}+\lambda x)}. \tag{4.8.23}$$

Note here that

$$\begin{aligned} h_m^+(\rho') &\le h_\infty^+(0) = \int_0^\infty d\lambda \int_0^\infty dx\, (\alpha/b)\, x^{-\alpha/b}\exp\{-\left(\Gamma(1-\alpha/b)\lambda^{\alpha/b}+\lambda x\right)\} \\ &= (\alpha/b)\int_0^\infty d\lambda\, \lambda^{\alpha/b-1}\Gamma(1-\alpha/b)\exp\{-\left(\Gamma(1-\alpha/b)\lambda^{\alpha/b}\right)\} = 1. \end{aligned} \tag{4.8.24}$$

By Fubini we thus may interchange the order of integration in (4.8.23), and setting for $x\ge 0$

$$df_{\alpha/b,m}(x) \equiv (\alpha/b)\, x^{-\alpha/b}\mathbb{1}_{\{x\ge 1/m\}}dx \int_0^\infty d\lambda \exp\{-\left(\Gamma(1-\alpha/b)\lambda^{\alpha/b}+\lambda x\right)\}, \tag{4.8.25}$$

we obtain

$$h_m^+(\rho') = \int_0^\infty \exp\{-\rho'/x^{(1-\theta)/b}\}df_{\alpha/b,m}(x). \tag{4.8.26}$$

Again by (4.8.24) dominated convergence applies and passing to the limit $n\to\infty$ in (4.8.26) yields (4.8.10). The proof of (4.8.2) is now complete. $\square$

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